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ON PRIMAL IDEALS

LADISLAS FUCHS

1. Introduction. In my paper *On quasi-primary ideals*[1] I have added a fifth decomposition of an ideal to those given by E. Noether.[2] Thus in a commutative ring with unit element in which the maximal condition is satisfied there are in general five distinct decompositions of an ideal; namely as shortest[3] intersection of a finite set of (i) *irreducible*, (ii) *primary*, (iii) *quasi-primary*, (iv) *relatively-prime-indecomposable*, (v) *direct-indecomposable ideals*. The components in (iv) and in (v) are unique, while in (i), (ii), (iii) the associated prime ideals, but not necessarily the components themselves, are unique. In (i) one prime ideal may belong to more than one component, but always to the same number of components. Each prime ideal that occurs in (i) occurs even in (ii) but with a single multiplicity, while in (iii) only the *minimal* prime ideals of (ii) occur, each but once.

The comparison of the features of the prime ideals associated with components of (ii) and of (iii) suggests that, perhaps, a new type of decomposition may be defined, one whose components are associated with just the *maximal* prime ideals of (ii). The sought type of ideals will be realised by those which coincide with their single "principal component" in the sense of W. Krull.[4] These ideals will be defined here more directly, again by making use of the familiar concept of "relative primeness." However, it must be emphasized that the method used here is different from Krull's method. Moreover it seems quite questionable whether all principal components are primal.

2. Definition and preliminaries. We shall say that $\mathfrak{q}$ is a *primal ideal* if the elements which are not prime[5] to it form an ideal $\mathfrak{p}$ called

Received by the editors June 15, 1948 and, in revised form, August 15, 1948.

[1] Acta Univ. Szeged. vol. 11 (1947) pp. 174–183. An ideal is quasi-primary if no product of two elements belongs to it unless a power of one of them belongs to it, or, otherwise, if its radical is prime.

[2] E. Noether, *Idealtheorie in Ringbereichen*, Math. Ann. vol. 83 (1921) pp. 24–66.

[3] The term "shortest" means that no component may simply be omitted and no intersection of more than one component is again of the same type.

[4] W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Math. Ann. vol. 101 (1929) pp. 729–744. If $\mathfrak{p}$ is a maximal prime divisor of $\mathfrak{a}$ (that is, $\mathfrak{p}$ is a maximal ideal that contains no element prime to $\mathfrak{a}$), then the principal component $\mathfrak{a}(\mathfrak{p})$ is defined to consist of all elements whose product by a properly chosen element not in $\mathfrak{p}$ belongs to $\mathfrak{a}$.

[5] a is called (relatively) prime to $\mathfrak{b}$, if $ab \in \mathfrak{b}$ is impossible unless $b \in \mathfrak{b}$; cf. Noether, loc. cit. footnote 2, p. 45.

the *adjoint ideal of* $\mathfrak{g}$. As bc together with b is not prime to $\mathfrak{g}$ whatever the element c, what we require is that together with two elements their difference shall be non-prime to $\mathfrak{g}$.

Evidently, $\mathfrak{g}$ is a primal ideal and $\mathfrak{p}$ is adjoint to $\mathfrak{g}$ if and only if $ab \in \mathfrak{g}$ and $b \notin \mathfrak{g}$ imply $a \in \mathfrak{p}$ and, conversely, whenever $a \in \mathfrak{p}$, there always exists an element b not in $\mathfrak{g}$ such that $ab \in \mathfrak{g}$.

The elements non-prime to $\mathfrak{g}$ represent in the residue class ring $\mathfrak{R}/\mathfrak{g}$ the zero factors, therefore the definition of primal ideals may also be formulated as follows: *in* $\mathfrak{R}/\mathfrak{g}$ *the zero factors form an ideal*, namely $\mathfrak{p}/\mathfrak{g}$.

The product of two elements prime to an ideal $\mathfrak{g}$ is again prime to it,[6] since, assuming b and c prime to $\mathfrak{g}$, $abc \in \mathfrak{g}$ implies $ab \in \mathfrak{g}$ and this implies $a \in \mathfrak{g}$. Therefore no product of two elements not belonging to $\mathfrak{p}$ belongs to $\mathfrak{p}$, thus *the adjoint ideal is prime.* This fact shows the analogy between the ideals adjoint to primal ideals and those associated with quasi-primary ideals.

The primal ideals may be considered as an extension of the idea of primary ideals. Indeed, the elements non-prime to a primary ideal are just the elements of its prime radical. But there are in general primal ideals which are not primary; moreover a primal ideal may be quasi-primary and need not be primary. To illustrate this, let us consider the ring of polynomials in x and y with rational numbers for their coefficients. In this ring $\mathfrak{x} = (x^2, xy)$ is a quasi-primary ideal with the radical (x) and is primal with (x, y) as adjoint prime ideal. $\mathfrak{x}$ is not primary for $xy \in \mathfrak{x}$ and neither x nor any power of y belongs to $\mathfrak{x}$. As another illustration we may refer to a valuation ring with a non-archimedian value-group,[7] where all ideals are *both* quasi-primary [for if $bc \in \mathfrak{a}$, then either $b^2 \in \mathfrak{a}$ or $c^2 \in \mathfrak{a}$ according as c divides b or b divides c] *and* primal [for if $b_1c_1 \in \mathfrak{a}$, $b_2c_2 \in \mathfrak{a}$ with $c_1 \notin \mathfrak{a}$, $c_2 \notin \mathfrak{a}$ and, for example, c_1 divides c_2, then $c_2(b_1 - b_2) \in \mathfrak{a}$], but not necessarily primary!

If, however, an ideal $\mathfrak{g}$ is quasi-primary and primal, and at the same time its prime radical and its adjoint prime ideal coincide, then $\mathfrak{g}$ is necessarily primary; for $ab \in \mathfrak{g}$, $b \notin \mathfrak{g}$ imply that a belongs to the adjoint ideal, that is, to the radical, therefore it follows that a certain power of a belongs to $\mathfrak{g}$: $\mathfrak{g}$ is indeed primary.[8]

One simple remark will be inserted here in order to make the con-

[6] With the terminology of Krull: the system of elements prime to $\mathfrak{g}$ is multiplicatively closed.

[7] W. Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. vol. 167 (1932) pp. 160–196.

[8] Cf. Krull's corollary to Satz 6, loc. cit. footnote 4, p. 737.

nection between primal ideals and principal components more apparent. Namely, we shall show that the principal component $\mathfrak{a}(\mathfrak{p})$ of $\mathfrak{a}$ belonging to the maximal prime ideal $\mathfrak{p}$ is a multiple of the intersection $\mathfrak{a}^*(\mathfrak{p})$ of all primal divisors of $\mathfrak{a}$ whose adjoint prime ideal is contained in $\mathfrak{p}$. Indeed, if $\mathfrak{g}$ is such a primal ideal and $a \in \mathfrak{a}(\mathfrak{p})$, then by definition there is an element b not in $\mathfrak{p}$ such that $ab \in \mathfrak{a}$. Now $ab \in \mathfrak{g}$, and this implies $a \in \mathfrak{g}$, for $b \notin \mathfrak{p}$ must be prime to $\mathfrak{g}$; therefore $\mathfrak{a}(\mathfrak{p}) \subseteq \mathfrak{g}$. It is, however, an open question whether $\mathfrak{a}^*(\mathfrak{p})$ is always equal to $\mathfrak{a}(\mathfrak{p})$ or not.

The great importance of primal ideals lies in the fact that the following theorem may be proved in complete generality, without making use of any condition.

THEOREM 1. *Every irreducible ideal is primal.*

The proof is based on a simple law on forming ideal quotients.[9] Let $\mathfrak{i}$ be any irreducible ideal and b_1, b_2 two elements which are not prime to $\mathfrak{i}$. Then $\mathfrak{i}:(b_1)$ and $\mathfrak{i}:(b_2)$ are proper divisors of $\mathfrak{i}$, hence their intersection $\mathfrak{i}:(b_1) \cap \mathfrak{i}:(b_2) = \mathfrak{i}:((b_1)+(b_2))$ cannot equal $\mathfrak{i}$, consequently, the element $b_1 - b_2$ cannot be prime to $\mathfrak{i}$, q.e.d.

3. **The intersection of primal ideals.** We now inquire when the intersection of primal ideals is again primal. It will seem no doubt somewhat surprising at first glance that the intersection of two primal ideals with the same adjoint prime ideal is not necessarily primal, not even in rings with basis theorem. For instance, in the polynomial domain considered above (x^2, xy) and (xy, y^2) are primal ideals both adjoint to (x, y), but they have for intersection a non-primal ideal, namely (xy).[10] But if we restrict ourselves merely to *reduced*[11] intersections, we may state the following theorem.

THEOREM 2. *The reduced intersection of a finite number of primal ideals* $\mathfrak{a} = \mathfrak{g}_1 \cap \cdots \cap \mathfrak{g}_n$ *with with* $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ *as adjoint prime ideals is again primal if and only if one prime ideal* $\mathfrak{p}_j$ *divides all other* $\mathfrak{p}_i$. *Then* $\mathfrak{p}_j$ *is adjoint to* $\mathfrak{a}$.

From the hypothesis that the intersection $\mathfrak{g}_1 \cap \cdots \cap \mathfrak{g}_n$ is reduced, we infer by a theorem of E. Noether[12] that b is not prime to $\mathfrak{a}$ if and only if it is not prime to at least one of $\mathfrak{g}_i$, that is, if and only if

[9] $\mathfrak{a}:(b_1+b_2) = \mathfrak{a}:b_1 \cap \mathfrak{a}b_2$. See, for example, B. L. van der Waerden, *Moderne Algebra*, vol. 2, 2d ed., Berlin, 1940, p. 25.

[10] x and y are not prime to (xy), but $x-y$ is prime to it.

[11] "Reduced" means that no component may be replaced by one of its proper divisors.

[12] See Noether, loc. cit. footnote 2, Satz X, p. 45.

b belongs to at least one of the $\mathfrak{p}_i$. This tells us that $\mathfrak{a}$ is primal if and only if the union of the elements of all $\mathfrak{p}_i$ is a prime ideal. Since the sufficiency of the stated condition is thus clear, to bring the proof to an end we must still show that the union of the elements of a finite set of primes is not a prime ideal again unless one divides all others.

Let $\mathfrak{p}_1^*, \cdots, \mathfrak{p}_k^*$ denote the different maximal ones of $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$, that is, those which are divided by no different one in the set of $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$; further assume $k>1$. None of $\mathfrak{p}_i^*$ divides another one; we can therefore find elements p_i $(i=1, \cdots, k)$ contained in $\mathfrak{p}_1^* \cap \cdots \cap \mathfrak{p}_{i-1}^* \cap \mathfrak{p}_{i+1}^* \cap \cdots \cap \mathfrak{p}_k^*$ but not in $\mathfrak{p}_i^*$. If $\mathfrak{a}$ were primal, the fact that the $\mathfrak{p}_i$ are not prime to $\mathfrak{a}$ would imply that $p=p_1+\cdots+p_k$ is not prime to $\mathfrak{a}$. Hence p would belong to one of the $\mathfrak{p}_i$ and so, a fortiori, to one of $\mathfrak{p}_i^*$. This is, however, a contradiction, since each p_m except p_i belongs to $\mathfrak{p}_i^*$. The proof is thus completed.

As an immediate consequence of Theorem 2 we may establish that in a ring with maximal condition an ideal is primal if and only if one of the prime ideals associated with its representation (ii) contains all the others. The assertion becomes evident in view of Theorem 2 if one takes into account that the same prime ideals are associated with representation (i) as with (ii) and, further, that a shortest irreducible representation is at the same time reduced.[13] This fact implies that if $\mathfrak{p}$ adjoint to the primal ideal $\mathfrak{q}$ is a minimal prime ideal in a ring with maximal condition, then $\mathfrak{q}$ is primary, for its associated prime ideals are multiples of $\mathfrak{p}$ and so necessarily equal to $\mathfrak{p}$.[14]

4. The existence of primal decompositions. Now we shall be concerned with decompositions into primal ideals. The bare existence of primal decompositions is all that we can prove in general.

THEOREM 3. *Every ideal is representable as the intersection of its primal divisors.*

Since the unit ideal $\mathfrak{o}$ is primal and divides each ideal, the intersection of the primal divisors of an arbitrary ideal $\mathfrak{a}$ exists and, what is quite evident, divides $\mathfrak{a}$. What we have thus to prove is that if a does not belong to $\mathfrak{a}$, there is a primal divisor of $\mathfrak{a}$ which does not contain a. By Zorn's well known lemma[15] we may conclude that there exists an ideal $\mathfrak{q}$ dividing $\mathfrak{a}$ but not containing a such that each proper

[13] Cf. Noether, loc. cit. footnote 2, Hilfssatz II, p. 35.

[14] The intersection of a finite number of primary ideals associated with the same prime ideal is primary again; see, for example, van der Waerden, loc. cit. footnote 9, p. 32.

[15] M. Zorn, *A remark on method in transfinite algebra*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 667–670.

divisor of $\mathfrak{g}$ contains a. Let b_1, b_2 be elements not prime to $\mathfrak{g}$ and determine c_1, c_2 not in $\mathfrak{g}$ such that $b_1c_1 \in \mathfrak{g}$, $b_2c_2 \in \mathfrak{g}$. The ideals $\mathfrak{g}+(c_1)$ and $\mathfrak{g}+(c_2)$ are proper divisors of $\mathfrak{g}$, therefore both contain a, $a \in \mathfrak{g}+(c_1)$ and $a \in \mathfrak{g}+(c_2)$. We hence have $ab_1 \in \mathfrak{g}b_1+(b_1c_1) \subset \mathfrak{g}$ and similarly $ab_2 \in \mathfrak{g}$, so that one may see that b_1-b_2 is not prime to $\mathfrak{g}$, for, while $a(b_1-b_2) \in \mathfrak{g}$, a itself does not belong to $\mathfrak{g}$.[16]

5. **Finite decompositions.** In order to analyze the finite primal decompositions, instead of confining ourselves merely to rings with ascending chain condition we shall consider somewhat more general rings. We want to establish, as far as we can, the existence of a decomposition into a finite set of primal ideals and even the unicity of the prime ideals adjoint to the components. It appears to be necessary to restrict ourselves to rings in which every ideal is representable as the intersection of a finite number of irreducible ideals as well as to normal representations.[17] Our stated problem is completely solved only in such rings and for such decompositions.

Given any ideal $\mathfrak{a}$, we start with one of its finite shortest, and so necessarily reduced, irreducible representations and unite all components whose adjoint prime ideal $\mathfrak{p}$ is the same as or is a multiple of an arbitrary maximal $\mathfrak{p}_j$, say, of $\mathfrak{p}_1$. Then, by Theorem 2, we obtain a new primal ideal with $\mathfrak{p}_1$ as adjoint prime ideal. Thus proceeding, we finally arrive at a shortest primal decomposition which will be reduced[18] and hence normal.

THEOREM 4. *In a ring where each ideal may be represented as the intersection of a finite set of irreducible ideals, each ideal possesses a normal decomposition into a finite number of primal ideals.*

6. **The unicity statement.** Finally, we establish the unicity of the

[16] I am indebted to Professor Nathan Jacobson for having called my attention to N. H. McCoy's paper, *Subdirectly irreducible commutative rings*, Duke Math. J. vol. 12 (1945) pp. 381–387, whose Theorem 1 includes Theorem 3 of the present paper. The residue class ring $\mathfrak{R}/\mathfrak{g}$ with respect to the ideal $\mathfrak{g}$ constructed in the proof of Theorem 3 is subdirectly irreducible in the sense of G. Birkhoff (Bull. Amer. Math. Soc. vol. 50 (1944) pp. 761–768); further, $\mathfrak{R}/\mathfrak{a}$ is the subdirect union of the subdirectly irreducible rings $\mathfrak{R}/\mathfrak{g}$. McCoy's cited theorem (especially part (ii)) asserts that in $\mathfrak{R}/\mathfrak{g}$ the divisors of zero form an ideal, namely $\mathfrak{p}/\mathfrak{g}$, that is, $\mathfrak{g}$ is primal. (McCoy's Theorem 2 corresponds to the trivial case where 0 is adjoint to $\mathfrak{g}$).

[17] We use the term "normal" to mean representations which are shortest and reduced.

[18] Noether has proved (loc. cit. footnote 2, pp. 36–37) that if all components may be represented as the intersection of a finite number of irreducible ideals, then the reducedness does not alter during the procedure of joining some components into one ideal.

adjoint prime ideals of normal primal decompositions.[19]

If $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_n$ and $\mathfrak{a}=\mathfrak{q}_1^*\cap\cdots\cap\mathfrak{q}_m^*$ are two normal primal decompositions with the adjoint prime ideals $\mathfrak{p}_1,\cdots,\mathfrak{p}_n$ and $\mathfrak{p}_1^*,\cdots,\mathfrak{p}_m^*$ respectively, we first show that each $\mathfrak{p}_i$ $(i=1,\cdots,n)$ is divided by one of $\mathfrak{p}_j^*$ $(j=1,\cdots,m)$ and each $\mathfrak{p}_j^*$ by one of $\mathfrak{p}_i$. For if $\mathfrak{p}_1$ were divided by no one of $\mathfrak{p}_j^*$, we should get a contradiction as follows. The hypothesis admits of choosing an element p_j in $\mathfrak{p}_1\cap\mathfrak{p}_1^*\cap\cdots\cap\mathfrak{p}_{j-1}^*\cap\mathfrak{p}_{j+1}^*\cap\cdots\cap\mathfrak{p}_m^*$ (for $j=1,\cdots,m$) such that p_j does not belong to $\mathfrak{p}_j^*$. Then $p=p_1+\cdots+p_m$ will belong to $\mathfrak{p}_1$ but to none of $\mathfrak{p}_j^*$, a contradiction, since p is not prime to $\mathfrak{a}$, and therefore, if we use Theorem 2, it must belong to one of $\mathfrak{p}_j^*$.

If now $\mathfrak{p}_1^*$ divides $\mathfrak{p}_1$ and is divided by $\mathfrak{p}_i$, then $\mathfrak{p}_i$ divides $\mathfrak{p}_1$; this is impossible unless $\mathfrak{p}_i=\mathfrak{p}_1$. Hence we get $\mathfrak{p}_1=\mathfrak{p}_1^*$ and the theorem that we are going to formulate is proved.

THEOREM 5. *In two finite normal primal decompositions of an ideal the numbers of the components as well as their adjoint prime ideals are necessarily the same.*[20]

7. **An illustration.** An example will establish that the primal decomposition coincides in general with no other one of (i)–(v). In the polynomial ring of x, y, z decompositions (ii) and (iii) of the ideal $\mathfrak{w}=(x^2y,\ xy^2z^2)$ are:

(ii) $$\mathfrak{w}=(x)\cap(y)\cap(x^2,\ y^2)\cap(x^2,\ z^2)$$

with (x), (y), $(x,\ y)$, $(x,\ z)$ as associated prime ideals;

(iii) $$\mathfrak{w}=(x^2,\ xy^2z^2)\cap(y)$$

with the radicals (x), (y).

A normal primal decomposition of $\mathfrak{w}$ is $\mathfrak{w}=(x^2y,\ xy^2)\cap(x^2,\ z^2)$, the adjoint prime ideals being $(x,\ y)$ and $(x,\ z)$. It is interesting that in this case there exists a shortest quasi-primary decomposition which is at the same time normal primal, namely $\mathfrak{w}=(x^2,\ xz^2)\cap(x^2y,\ y^2)$ with the radicals (x), (y) and with the adjoint prime ideals $(x,\ z)$, $(x,\ y)$.

Finally, $\mathfrak{w}$ is a relatively-prime-indecomposable ideal.

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[19] Without reducedness the statement is not true. For example, $(xy)=(x)\cap(y)=(x^2,\ xy)\cap(y)=(x)\cap(xy,\ y^2)=(x^2,\ xy)\cap(xy,\ y^2)$ where the adjoint prime ideals are (x), (y); $(x,\ y)$, (y); (x), $(x,\ y)$; $(x,\ y)$, $(x,\ y)$.

[20] On using the unicity of the associated prime ideals of decomposition (i), Theorem 5 is evident in rings with basis theorem.

ON A THEOREM OF ERDÖS AND TURÁN

ALFRED RÉNYI

Let $p_1=2$, $p_2=3$, p_3, $\cdots$, p_n, $\cdots$ denote the sequence of primes. P. Erdös and P. Turán proved recently,[1] among a series of similar results, that the sequence $\log p_n$ is neither convex nor concave from some large n onwards, that is, that the sequence

$$p_{n+1}\cdot p_{n-1}-p_n^2$$

changes its sign an infinite number of times. The aim of the present paper is to prove an improvement of this theorem. It shall be proved that the total curvature of the polygonal line having the vertices $n+i\cdot\log p_n$ $(n=1, 2, 3, \cdots)$ in the complex z-plane is infinite. This implies that $p_{n+1}p_{n-1}-p_n^2$ changes its sign an infinite number of times, but our result also contains some quantitative information regarding the oscillations of the sequence $p_{n+1}p_{n-1}-p_n^2$.

The total curvature G of a finite polygonal line situated in the complex z-plane may be defined as follows:

$$G=\sum_{k=2}^{n-1}\left|\arg\frac{z_{k+1}-z_k}{z_k-z_{k-1}}\right| \tag{1}$$

where $z_1, z_2, \cdots, z_n$ denote the complex numbers corresponding to the vertices of the polygonal line in the complex z-plane. The total curvature of an infinite polygonal line is defined as the limit of the total curvature of its segments, provided this limit exists, and as ∞ in case the total curvature of its segments tends to infinity. It is easy to see that if the polygonal line Π having the vertices $n+i\cdot\log p_n$ were convex or concave from some point onwards, its total curvature would be finite. It will be shown that this is not so, as we shall prove the following theorem.

THEOREM. *The total curvature, as defined by* (1), *of the polygonal line* Π_N *having the vertices* $z_n=n+i\cdot\log p_n$ $(p_n\leqq N)$ *exceeds* $c\cdot\log\log\log N$, *where* c *is a positive constant.*

Let $\pi(n)$ denote the number of primes not exceeding n. For proving our theorem, we shall need only the following propositions of the prime number theory:

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[1] P. Erdös and P. Turán, *On some new questions on the distribution of prime numbers*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 371–378.

I. $\pi(n) \sim \dfrac{n}{\log n}$, II. $\log\left(\prod_{p \leqq n} p\right) \sim n$,

III. $\lim_{n \to \infty} \dfrac{p_{n+1}}{p_n} = 1$, IV. $p_{n+1} - p_n = O\left(\dfrac{p_n}{\log^2 p_n}\right)$.

I and II are different forms of the prime number theorem, III is a consequence of the prime number theorem, and IV follows from the de La Vallée Poussin estimation of the remainder term of the prime number formula. (In fact by a slight modification of the argument we could dispense with the prime number theorem and use only more elementary results.)

Let G_N denote the total curvature of π_N. We have

$$G_N = \sum_{p_{n+1} \leqq N} \left| \arctan \log \frac{p_{n+1}}{p_n} - \arctan \log \frac{p_n}{p_{n-1}} \right|. \tag{2}$$

In what follows $c_1, c_2, \cdots$ denote positive constants. Using III and the identity $\tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1+\tan\alpha \tan\beta)$, we obtain from (2)

$$G_N \geqq c_1 \cdot \sum_{p_{n+1} \leqq N} \left| \log \frac{p_{n+1} \cdot p_{n-1}}{p_n^2} \right|. \tag{3}$$

Now we need the following lemma.

LEMMA A.

$$\sum_{n=1}^{\infty} \frac{(p_{n+1} - p_n)(p_n - p_{n-1})}{p_n^2} < \infty.$$

PROOF. According to IV, we have

$$\sum_{N \leqq p_n < 2N} \frac{(p_{n+1} - p_n)(p_n - p_{n-1})}{p_n^2} \leqq \max_{N \leqq p_n < 2N} \left(\frac{p_{n+1} - p_n}{p_n^2} \right) \cdot \sum_{N \leqq p_n < 2N} (p_n - p_{n-1}) < \frac{c_2}{\log^2 N}. \tag{4}$$

Putting $N = 2^k$ in (4) and summing for $k = 1, 2, 3, \cdots$, we obtain Lemma A.

When we apply Lemma A and the identity

$$\frac{p_{n+1} \cdot p_{n-1}}{p_n^2} = 1 + \frac{p_{n+1} - 2p_n + p_{n-1}}{p_n} - \frac{(p_{n+1} - p_n)(p_n - p_{n-1})}{p_n^2}, \tag{5}$$

it follows easily that

$$G_N \geqq c_3 \sum_{p_{n+1} \leqq N} \frac{|p_{n+1} - 2p_n + p_{n-1}|}{p_n} - c_4. \tag{6}$$

Now we prove the following lemma.

LEMMA B. *If each of*

$$a, a + D, a + 2D, \cdots, a + k \cdot D \tag{7}$$

is a prime, we have

$$D \geqq \exp(c_5 k). \tag{8}$$

PROOF. First we have $k < a$, for if not, $a + aD$, which is composite, would be contained in the sequence (7). Further if p is a prime, $p \leqq k$, p must be a divisor of D, for if not, the numbers a, $a+D$, $a+2D, \cdots, a+(p-1)\cdot D$ would be all incongruent mod p, and thus one of them would be divisible by p, and not equal to p owing to $p \leqq k < a$. Thus D is divisible by p, and when we apply II, Lemma B follows.

To prove our theorem we need one more lemma. For the sake of brevity let us put

$$\Delta_k = p_{k+1} - 2p_k + p_{k-1}. \tag{9}$$

Let us divide the sequence of primes $p_{k+1} \leqq n$ into blocks $(p_{\nu_r}, p_{\nu_r+1}, p_{\nu_r+2}, \cdots, p_{\nu_r+k_r})$ characterized by $\Delta_{\nu_r} \neq 0$, $\Delta_{\nu_r+\nu} = 0$ for $\nu = 1, 2, \cdots, k_r$. The number of these blocks shall be denoted by $\lambda(n)$, that is, we put

$$\lambda(n) = \sum_{p_{\nu_r+1} \leqq n} 1. \tag{10}$$

We shall prove the following lemma.

LEMMA C.

$$\lambda(n) \geqq \frac{c_6 n}{\log n \cdot \log \log n}.$$

PROOF. According to Lemma B we have

$$\sum_{r=1}^{\lambda(n)} k_r \cdot \exp(c_5 k_r) \leqq \sum_{p_{k+1} \leqq n} (p_{k+1} - p_k) \leqq n. \tag{11}$$

On the other hand

$$\sum_{r=1}^{\lambda(n)} (k_r + 1) \geqq \pi(n) - 1. \tag{12}$$

If $\lambda(n)>\pi(n)/2$, our lemma is proved. Thus we may suppose $\lambda(n)<\pi(n)/2$. In this case it follows from (12) and I that

$$\sum_{r=1}^{\lambda(n)} k_r > \frac{c_7 \cdot n}{\log n}. \tag{13}$$

Applying the inequality of the arithmetic and geometric means as well as the inequality of Cauchy-Schwarz, we obtain, using (11) and (13), that

$$\begin{aligned} n &\geqq \sum_{r=1}^{\lambda(n)} k_r \exp(c_5 k_r) \geqq \left(\sum_{r=1}^{\lambda(n)} k_r\right) \exp\left(c_5 \frac{\sum_{r=1}^{\lambda(n)} k_r^2}{\sum_{r=1}^{\lambda(n)} k_r}\right) \\ &\geqq \frac{c_7 n}{\log n} \cdot \exp\left(\frac{c_5 \sum_{r=1}^{\lambda(n)} k_r}{\lambda(n)}\right). \end{aligned} \tag{14}$$

It follows that

$$\log\log n \geqq \log c_7 + \frac{c_5 c_7 n}{\lambda(n)\log n}, \tag{15}$$

which proves Lemma C.

Now the proof of our theorem can be easily completed. Using (6) and with respect to $\Delta_{\nu_r} \neq 0$ we have evidently

$$G_N \geqq c_3 \sum_{r=1}^{\lambda(N)} \frac{1}{p_{\nu_r}} - c_4. \tag{16}$$

Hence by (10)

$$\begin{aligned} G_N &\geqq c_3 \sum_{n=1}^{N} \frac{\lambda(n)-\lambda(n-1)}{n} - c_4 \geqq c_3 \sum_{n=1}^{N} \frac{\lambda(n)}{n(n+1)} - c_4 \\ &\geqq c_8 \sum_{n=1}^{N} \frac{1}{n \cdot \log n \cdot \log\log n}, \end{aligned} \tag{17}$$

which proves our theorem.

It may be mentioned that the question whether there exist blocks of primes $(p_{\nu_r}, p_{\nu_r+1}, \cdots, p_{\nu_r+k_r})$ with $\Delta_{\nu_r+\nu}=0$, $\nu=1, 2, \cdots, k_r$, of any length k_r, and even the question whether there exist an infinity of indices k for which $\Delta_k=0$ are unsolved.

UNIVERSITY OF BUDAPEST

NOTE ON MAXIMAL ALGEBRAS

G. HOCHSCHILD

Introduction. It has been shown in a previous paper [3][1] that every algebra A with radical R, such that A/R is separable, is a homomorphic image of a certain maximal algebra which is determined to within an isomorphism by A/R, the A/R-module (two-sided) R/R^2, and the index of nilpotency of R. Furthermore, some indication was given of how the structure of maximal algebras can be determined in simple cases.

Here, we wish to give a further illustration by describing a rather wide class of maximal and primary algebras whose structure will be shown to resemble that of crossed products, in certain respects. In fact, we shall impose a certain normality condition and then trace the consequences of a few simple facts of the noncommutative Galois theory.

An algebra B over the field F, with radical R, is called primary if it has an identity element and if B/R is simple. As is well known,[2] B is then isomorphic with a Kronecker product $F_m \times C$, where F_m denotes the full matrix algebra of degree m over F, and where C is completely primary, in the sense that it has an identity element and that the quotient of C by its radical is a division algebra over F. We are concerned with primary algebras B for which this division algebra (which is determined to within an isomorphism by B) is normal over F, in the sense of the noncommutative Galois theory.[3] This will be the case if and only if the center Z of B/R is a separable normal extension field of F and every automorphism of Z over F is induced by an automorphism of B/R.

A completely primary algebra C with radical S will henceforth be called quasinormal if C/S is normal over F. If ϕ is an isomorphism of $F_m \times C$ onto B then ϕ maps the radical $F_m \times S$ onto the radical R of B, and B/R is isomorphic with $F_m \times C/S$. Therefore, if C is quasinormal over F, then B/R is automatically separable over F. By [3], B has then a maximal related extension B^* which is evidently primary. Moreover, it is easily seen that $B^* \approx F_m \times C^*$, where C^* is the maximal related extension of C, and is quasinormal over F. Finally, the natural extension to B^* of a homomorphism of C^* onto C is a homomorphism of B^* onto B. From these facts it is evident

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] See, for instance, [2, chap. 2, §9].

[3] See [1] and [5].

that, without loss of generality, we may confine our attention to quasinormal algebras.

1. **Representation in the radical.** Let B be quasinormal over F, and let R be its radical. By Wedderburn's theorem and the quasinormality we have $B=A+R$, where A is a normal division algebra over F. Moreover, it is easy to see that the identity element of A coincides with the identity element of B. Hence, if R is regarded in the natural fashion as a two-sided A-module, the identity element of A acts as the identity operator on either side.

As in [3, §4], we make a direct decomposition of the A-module R in the form $R=T+R^2$. The submodule T (which is isomorphic with the A-module R/R^2) is decomposed further into a direct sum of (two-sided) simple submodules. Now it is easily seen that every simple A-module over F is an operator homomorphic image of the Kronecker product $A\times A$ with respect to F, regarded in the natural fashion as a two-sided A-module. By applying either the noncommutative Galois theory, or the classical representation theory for simple algebras in conjunction with the theory of field composites (for $Z\times Z$, where Z is the center of A), we conclude that the simple components of the A-module $A\times A$ are of dimension 1 over A.[4] It follows that every simple A-module is of dimension 1 over A, and if u is a generator we have $u\cdot a=\sigma\{a\}\cdot u$, for every $a\in A$, where σ is an automorphism of A over F. Two such simple modules are operator isomorphic if and only if the corresponding automorphisms differ only by an inner automorphism of A.

It follows that there is a set of automorphisms $\sigma_1, \cdots, \sigma_s$ (not necessarily distinct) of A over F which are determined uniquely up to inner automorphisms,[5] and a corresponding decomposition, $T=A\cdot u_1+\cdots+A\cdot u_s$ of T into simple submodules, such that, for each index i and every $a\in A$, $u_i\cdot a=\sigma_i\{a\}\cdot u_i$.

2. **The standard maximal algebras.** Let B be as in §1. We construct a maximal related extension of B from A and T by the process described in [3, §4]. If B is already maximal the resulting "standard algebra" will be isomorphic with B.

The radical of the algebra to be constructed is the linearly direct sum of the "powers" $T^{(k)}$, $k=1, \cdots, n$, where n is the index of nil-

[4] A proof of this, in a terminology close to the present one, will be found in [4, §4].

[5] This because the simple components of a semisimple module are unique to within an automorphism of the whole module, as follows, for instance, from the evident fact that they are isomorphic images of the factors in a composition series.

potency of R. We have $T^{(1)}=T$, and generally $T^{(k+1)}$ is the Kronecker product $T^{(k)}\times T$, which means that in the elementary products $u\times v$, where $u\in T^{(k)}$ and $v\in T$, we identify $u\cdot a\times v$ with $u\times a\cdot v$ for every $a\in A$, and define the module operations such that $a\cdot(u\times v)=a\cdot u\times v$ and $(u\times v)\cdot a=u\times v\cdot a$. There is a natural multiplication $(T^{(k)}, T^{(l)})\rightarrow T^{(k+l)}$, for $k+l\leqq n$, and $(T^{(k)}, T^{(l)})\rightarrow(0)$ for $k+l>n$, by means of which the multiplication in the standard algebra is defined.

It is easy to see that the partial products $A\cdot u_i\times A\cdot u_j$ are simple A-modules and may be represented in the form $A\cdot u$, with $u\cdot a=\sigma_i\sigma_j\{a\}\cdot u$, for every $a\in A$. From this and §4 of [3], it can be seen immediately that the standard algebra may be described as follows:

Its elements are polynomials of total degree not greater than n, with coefficients in A, in the freely noncommuting "variables" $u_1, \cdots, u_s$. The elements are multiplied like ordinary polynomials, subject to the commutation rules $u_i a=\sigma_i\{a\}u_i$, and all the terms of degree greater than n are omitted. We denote this algebra by $A(u_1, \cdots, u_s)_n$. Our result is the following:

THEOREM. *Every quasinormal maximal algebra B with radical R is isomorphic with a standard "polynomial algebra" $B/R(u_1, \cdots, u_s)_n$, where s is the number of simple components of the B/R-module R/R^2, and n is the index of nilpotency of R. To each u_i there belongs an automorphism σ_i of B/R over F such that $u_i a=\sigma_i\{a\}u_i$ for every $a\in B/R$. These automorphisms are determined by B to within inner automorphisms of B/R. Two standard algebras with the same coefficient rings B/R are isomorphic if and only if their automorphisms σ_i and τ_i can be so indexed that, for each i, $\sigma_i\tau_i^{-1}$ is an inner automorphism of B/R.*

As in [3, §6], we obtain the following corollary:

COROLLARY. *If B is quasinormal and if R/R^2 is a simple B/R-module, then B is maximal and is isomorphic with a standard algebra $B/R(u_s)_n$. Two such algebras are isomorphic if and only if the corresponding automorphisms differ only by an inner automorphism of B/R.*

In general, every primary algebra satisfying the normality condition laid down in the introduction is a homomorphic image of an algebra of the type $F_m\times A(u_1, \cdots, u_s)_n$, the kernel of the homomorphism consisting of "polynomials" in which the "constant" terms and the terms of degree 1 in the u_i's are absent.

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UNIVERSITY OF ILLINOIS

ON THE RADICAL OF A LIE ALGEBRA

HARISH-CHANDRA

Let $\mathfrak{L}$ be a Lie algebra over a field K of characteristic zero. For any $X \in \mathfrak{L}$ we denote, as usual, the linear mapping $Y \to [X, Y]$ of $\mathfrak{L}$ into itself by ad X. Let Γ be the radical of $\mathfrak{L}$. Consider the set $\mathfrak{N}$ consisting of all $N \in \Gamma$ such that ad N is nilpotent. It was shown in a recent paper[1] that $\mathfrak{N}$ is the unique maximal nilpotent ideal[2] of $\mathfrak{L}$. Further if D is a derivation of Γ then $D\Gamma \subset \mathfrak{N}$.

For any $X, Y, Z \in \mathfrak{L}$ put $B(X, Y) = sp(\text{ad } X \text{ ad } Y)$ and $T(X, Y, Z) = sp(\text{ad } [X, Y] \text{ ad } Z)$. Then $B(X, Y)$ is a symmetric bilinear form on $\mathfrak{L}$ while $T(X, Y, Z)$ is a skewsymmetric trilinear form. It is easily verified that they are both invariant under all derivations of $\mathfrak{L}$, that is,

$$B(DX, Y) + B(X, DY) = 0,$$

$$T(DX, Y, Z) + T(X, DY, Z) + T(X, Y, DZ) = 0$$

for any derivation D and $X, Y, Z \in \mathfrak{L}$.

An ideal $\mathfrak{M}$ in $\mathfrak{L}$ is called characteristic if $D\mathfrak{M} \subset \mathfrak{M}$ for every derivation D of $\mathfrak{L}$. Our first theorem may now be stated as follows:

THEOREM 1. *An element X of $\mathfrak{L}$ belongs to the radical Γ if and only if $T(X, Y, Z) = 0$ for all $Y, Z \in \mathfrak{L}$.*[3]

As an immediate corollary we get the following:

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[1] Ann. of Math. vol. 50 (1949) p. 68.

[2] My attention has been drawn to a paper by Malcev (Bull. Acad. Sci. URSS. vol. 9 (1945) pp. 329–356) where it is shown that $\mathfrak{N}$ is an ideal.

[3] Since $T(X, Y, Z) = -T(Z, Y, X)$ this condition is clearly equivalent to $B(X, Y) = 0$ for all $Y \in \mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}]$. Professor Jacobson has kindly brought it to my notice that this theorem is contained in Cartan's thesis p. 109.

COROLLARY 1. *Both* Γ *and* $\mathfrak{N}$ *are characteristic ideals in* $\mathfrak{L}$.

For every $N \in \mathfrak{N}$, ad N is a nilpotent derivation of $\mathfrak{L}$. Hence $\sigma_N = \exp(\text{ad } N)$ is defined and is an automorphism of $\mathfrak{L}$. Let $\mathfrak{M}$ be any nilpotent ideal in $\mathfrak{L}$. Then $\mathfrak{M} \subset \mathfrak{N}$. By $G_{\mathfrak{M}}$ we denote the group of all automorphisms of $\mathfrak{L}$ of the form $\sigma_{M_1}\sigma_{M_2}\cdots\sigma_{M_r}$ where $M_1, \cdots, M_r \in \mathfrak{M}$ and $r \geq 1$. Clearly every ideal is invariant under $G_{\mathfrak{M}}$. Our second theorem now runs as follows:

THEOREM 2. *Let* $\mathfrak{S}$ *be a semisimple subalgebra of* $\mathfrak{L}$ *such that* $\mathfrak{L} = \mathfrak{S} + \Gamma$. *Then, given any semisimple subalgebra* $\mathfrak{M}$ *of* $\mathfrak{L}$, *there exists a* $\sigma \in G_{\mathfrak{N}}$ *such that* $\mathfrak{M} \subset \sigma\mathfrak{S}$.

The following two corollaries follow immediately from this theorem.

COROLLARY 2. *Any maximal semisimple subalgebra of* $\mathfrak{L}$ *is isomorphic to* $\mathfrak{L}/\Gamma$.

COROLLARY 3. *Given any two maximal semisimple subalgebras* $\mathfrak{S}_1, \mathfrak{S}_2$ *of* $\mathfrak{L}$, *there exists a* $\tau \in G_{\mathfrak{N}}$ *such that* $\tau\mathfrak{S}_1 = \mathfrak{S}_2$.

Corollary 3 is a sharper form of a result due to Malcev.[4]

PROOF OF THEOREM 1. First we shall prove that for any $N \in \mathfrak{N}$, $B(N, Z) = 0$ for all $Z \in \mathfrak{L}$. For any $s \geq 1$ define $\mathfrak{N}_{(s)}$ by induction as follows. $\mathfrak{N}_{(1)} = \mathfrak{N}$, $\mathfrak{N}_{(s+1)} = [\mathfrak{N}, \mathfrak{N}_{(s)}]$. Then $\mathfrak{N}_{(s)}$ is an ideal in $\mathfrak{L}$ and therefore (ad N ad Z)$\mathfrak{L} \subset \mathfrak{N}$ and (ad N ad Z)$\mathfrak{N}_{(s)} \subset \mathfrak{N}_{(s+1)}$. Hence (ad N ad Z)$^{s+1}\mathfrak{L} = \mathfrak{N}_{(s)}$. But $\mathfrak{N}$ is nilpotent and therefore $\mathfrak{N}_{(s)} = \{0\}$ for some s. Hence (ad N ad Z)$^{s+1}\mathfrak{L} = \{0\}$ or (ad N ad Z)$^{s+1} = 0$. Therefore ad N ad Z is nilpotent and sp(ad N ad Z) $= B(N, Z) = 0$.

Let $\mathfrak{M}$ be the set of all $X \in \mathfrak{L}$ such that $T(X, Y, Z) = 0$ for all $Y, Z \in \mathfrak{L}$. Since T is invariant under all derivations of $\mathfrak{L}$, $\mathfrak{M}$ is a characteristic ideal in $\mathfrak{L}$. We have to show that $\mathfrak{M} = \Gamma$. Let $X \in \Gamma$. For any $Y \in \mathfrak{L}$, ad Y is a derivation of $\mathfrak{L}$ and Γ is invariant under ad Y. Hence ad Y induces a derivation of Γ and therefore (ad Y)$\Gamma \subset \mathfrak{N}$. So $[X, Y] = -(\text{ad } Y)X \in \mathfrak{N}$. Hence $B([X, Y], Z) = T(X, Y, Z) = 0$ for all $Z \in \mathfrak{L}$. Since this is true for every Y, $X \in \mathfrak{M}$. Hence $\Gamma \subset \mathfrak{M}$. On the other hand let $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{M}]$. Then it is clear that for any $M \in \mathfrak{M}'$, $B(M, Z) = 0$ for all $Z \in \mathfrak{L}$. Hence by Cartan's criterion for solvability $\mathfrak{M}'$ is solvable. Hence $\mathfrak{M}$ is solvable and therefore $\mathfrak{M} \subset \Gamma$. So the theorem is proved.

Since $\mathfrak{M}$ is a characteristic ideal the same is true of Γ. Hence if D is a derivation of $\mathfrak{L}$, $D\Gamma \subset \Gamma$ and so D induces a derivation of Γ. Therefore $D\Gamma \subset \mathfrak{N}$ and so $D\mathfrak{N} \subset D\Gamma \subset \mathfrak{N}$. Hence $\mathfrak{N}$ is also a characteristic ideal.

[4] A. Malcev, C. R. Acad. Sci. URSS. vol. 36 (1942) p. 42.

PROOF OF THEOREM 2. Since $\mathfrak{S}$ is semisimple $\mathfrak{S}\cap\Gamma=\{0\}$. Hence every $P\in\mathfrak{M}$ can be written uniquely as

$$P = S(P) + \nu(P)$$

where $S(P)\in\mathfrak{S}$ and $\nu(P)\in\mathfrak{N}$. Hence

$$[P, Q] = [S(P), S(Q)] + [S(P), \nu(Q)] + [\nu(P), S(Q)] + [\nu(P), \nu(Q)].$$

Therefore

$$S([P, Q]) = [S(P), S(Q)],$$
$$\nu([P, Q]) = [\nu(P), S(Q)] + [S(P), \nu(Q)] + [\nu(P), \nu(Q)].$$

We have already seen that (ad X)$\Gamma\subset\mathfrak{N}$ for any $X\in\mathfrak{L}$. Hence, $\nu([P, Q])\in\mathfrak{N}$. Let ν denote the mapping $P\to\nu(P)$. Then $\nu([\mathfrak{M}, \mathfrak{M}])\subset\mathfrak{N}$. But $\mathfrak{M}$ is semisimple. Therefore $[\mathfrak{M}, \mathfrak{M}]=\mathfrak{M}$ and $\nu(\mathfrak{M})\subset\mathfrak{N}$. Hence $\mathfrak{M}\subset\mathfrak{S}+\mathfrak{N}$.

Put $\mathfrak{L}_0=\mathfrak{S}+\mathfrak{N}$. First suppose that $\mathfrak{N}$ is abelian. The mapping $P\to S(P)$ is a homomorphic mapping of $\mathfrak{M}$ into $\mathfrak{S}$. For any $S\in\mathfrak{S}$ let D_S denote the derivation of $\mathfrak{N}$ given by $D_SN=[S, N]$ $(N\in\mathfrak{N})$. Then the mapping ρ defined by $\rho(P)=D_{S(P)}$ is a representation of $\mathfrak{M}$. Also, since $\mathfrak{N}$ is abelian

$$\nu([P, Q]) = [S(P), \nu(Q)] - [S(Q), \nu(P)] = \rho(P)\nu(Q) - \rho(Q)\nu(P).$$

Hence ν is a Whitehead mapping of $\mathfrak{M}$ into $\mathfrak{N}$ with respect to the representation ρ. Therefore by the first Whitehead lemma[5] there exists an element $-N\in\mathfrak{N}$ such that $\nu(P)=-\rho(P)N=[N, S(P)]$. Hence

$$P = S(P) + \nu(P) = S(P) + [N, S(P)] = \exp(\text{ad } N)S(P)$$

since (ad $N)^2=0$, $\mathfrak{N}$ being abelian. Therefore $\mathfrak{M}\subset\sigma_N\mathfrak{S}$ and so the theorem is proved in this case.

Now consider the general case. Let $n=\dim\mathfrak{N}$. If $n\leqq 1$, $\mathfrak{N}$ is abelian and so the theorem is true. Hence we can assume $n>1$ and use induction on n. Further we can assume that $\mathfrak{N}$ is not abelian so that $\mathfrak{N}'=[\mathfrak{N}, \mathfrak{N}]\neq\{0\}$. Let $X\to\overline{X}$ denote the natural homomorphism of $\mathfrak{L}_0$ onto $\overline{\mathfrak{L}}_0=\mathfrak{L}_0/\mathfrak{N}'$. The radical of $\overline{\mathfrak{L}}_0$ is $\overline{\mathfrak{N}}=\mathfrak{N}/\mathfrak{N}'$ which is abelian. Let $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{S}}$ be the images of $\mathfrak{M}$ and $\mathfrak{S}$ respectively in $\overline{\mathfrak{L}}_0$. Then they are both semisimple and

[5] See Hochschild, Amer. J. Math. vol. 64 (1942) p. 677.

$$\overline{\mathfrak{M}} \subset \overline{\mathfrak{S}} + \overline{\mathfrak{N}}.$$

Since $\overline{\mathfrak{N}}$ is abelian it follows from the above proof that there exists an $\overline{N} \in \overline{\mathfrak{N}}$ such that

$$\overline{\mathfrak{M}} \subset \sigma_{\overline{N}} \overline{\mathfrak{S}}.$$

Let $N \in \overline{N}$ ($N \in \mathfrak{N}$). The complete inverse image of $\sigma_{\overline{\mathfrak{R}}} \overline{\mathfrak{S}}$ in $\mathfrak{L}_0$ is $\mathfrak{S}_1 + \mathfrak{N}'$ where $\mathfrak{S}_1 = \sigma_N \mathfrak{S}$. Hence

$$\mathfrak{M} \subset \mathfrak{S}_1 + \mathfrak{N}'.$$

Since $\mathfrak{N}$ is nilpotent, dim $\mathfrak{N}' <$ dim $\mathfrak{N} = n$. Hence the induction hypothesis is applicable to $\mathfrak{L}_1 = \mathfrak{S}_1 + \mathfrak{N}'$ and therefore there exists a $\sigma_1 \in G_{\mathfrak{N}'}$ such that $\mathfrak{M} \subset \sigma_1 \mathfrak{S}_1$. But $G_{\mathfrak{N}'} \subset G_{\mathfrak{R}}$ and therefore $\sigma = \sigma_1 \sigma_N \in G_{\mathfrak{R}}$ and $\mathfrak{M} \subset \sigma \mathfrak{S}$.

Now let $\mathfrak{S}^*$ be any maximal semisimple algebra of $\mathfrak{L}$. It follows from the above theorem that $\mathfrak{S}^* \subset \sigma \mathfrak{S}$ for some $\sigma \in G_{\mathfrak{R}}$. Since σ is an automorphism of $\mathfrak{L}$, $\sigma \mathfrak{S}$ is semisimple. Therefore $\mathfrak{S}^* = \sigma \mathfrak{S} \cong \mathfrak{S} \cong \mathfrak{L}/\Gamma$.

If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two maximal semisimple subalgebras of $\mathfrak{L}$, we can find τ_1, $\tau_2 \in G_{\mathfrak{R}}$ such that $\mathfrak{S}_1 = \tau_1 \mathfrak{S}$, $\mathfrak{S}_2 = \tau_2 \mathfrak{S}$. Then $\tau \mathfrak{S}_1 = \mathfrak{S}_2$ where $\tau = \tau_2 \tau_1^{-1} \in G_{\mathfrak{R}}$.

Summary. Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic zero and let Γ be its radical. It is proved that any $X \in \mathfrak{L}$ belongs to Γ if and only if $sp(\text{ad}\,[X, Y]\,\text{ad}\,Z) = 0$ for all $Y, Z \in \mathfrak{L}$. Here $X \to \text{ad}\,X$ is the adjoint representation of $\mathfrak{L}$. Further let $\mathfrak{N}$ be the maximal nilpotent ideal of $\mathfrak{L}$ and let $\mathfrak{S}$ and $\mathfrak{S}^*$ be any two maximal semisimple subalgebras of $\mathfrak{L}$. Then $\mathfrak{S} + \mathfrak{N} = \mathfrak{S}^* + \mathfrak{N}$ and $\mathfrak{S}$ and $\mathfrak{S}^*$ are conjugate in a certain strict sense.

INSTITUTE FOR ADVANCED STUDY

A PROBLEM OF P. A. SMITH

HSIEN-CHUNG WANG

In a paper in this Bulletin,[1] P. A. Smith has mentioned the problem whether in a non-abelian Lie group G there exists a non-countable proper subgroup everywhere dense in G. We can see that a negative answer to this problem is unlikely as the non-existence of such group implies the well known continuum hypothesis. It is the aim of the present short note to show that each separable, locally compact, nondiscrete metric group has a subgroup possessing the above properties.

Let G be an abstract group and S a subset of G. The least subgroup of G which contains S will be called the *group closure* of S and denoted by $\mathrm{gcl}(S)$. Evidently, $\mathrm{gcl}(S)$ consists of all the finite products of the elements of S and their inverses. It follows immediately that $S=\mathrm{gcl}(S)$ if and only if S forms a subgroup of G.

Suppose R to be a subset of G and p an element of G such that p does not belong to the group closure $\mathrm{gcl}(R)$ of R. Using Zorn's Theorem[2] which is equivalent to the Axiom of Choice, we can construct a subset H of G having the following properties:

(i) H forms a subgroup of G;

(ii) H contains R but does not contain p;

(iii) if t is an element of G not belonging to H, then p is contained in the group closure $\mathrm{gcl}(H\cup t)$ of the union $H\cup t$. In general, there exists more than one such subgroup H. We shall call each of them *a maximal subgroup including R but excluding p.*

Now let us consider a separable, locally compact, nondiscrete metric group G. We choose, in G, a countable everywhere dense subset R. The group closure $\mathrm{gcl}(R)$ of R is also countable. However, the group G, being nondiscrete, is a perfect space. Therefore, G must be non-countable, for otherwise it would be homeomorphic with the set of all rational numbers[3] which is not locally compact. It follows that there exists an element p of G which does not belong to $\mathrm{gcl}(R)$. We can construct a maximal subgroup H including R but excluding p. Evidently, H forms a proper, everywhere dense subgroup of G.

We shall show that H is non-countable. For this purpose, let us

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[1] P. A. Smith, *Everywhere dense subgroups of a Lie group*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 309–312.

[2] S. Leftschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942, p. 5.

[3] W. Sierpinski, *Introduction to general topology*, Toronto, 1934, p. 107.

consider equations of the form

$$p = z^{m_1} h_1 z^{m_2} h_2 \cdots h_s z^{m_{s+1}} \tag{1}$$

where m_i $(i=1, 2, \cdots)$ denote integers, h_i elements in H, and z the unknown. The set C of solutions of such an equation is closed in G, and from the maximal property of H, C belongs to the complement $G-H$ of H. Since H is everywhere dense, C is nowhere dense.

Suppose H to be countable. Then the aggregate of all equations of the form (1) is countable as well. Thus the union $Z=\cup C$ of all the possible C's is a set of the first category. Moreover, we can easily see that Z coincides with the complement $G-H$ of H. On the other hand, H is countable and G perfect, so that H is nowhere dense in G. It follows then that $G=H+Z$ is of the first category in itself. This contradicts the fact that G is a locally compact metric space. Therefore, H cannot be countable and we arrive at the following:

In any separable, locally compact, non-discrete metric group G, there always exists a non-countable proper subgroup filling G densely.[4]

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA

[4] The author wishes to express his sincere thanks to Professor D. Montgomery for his valuable suggestions which enable the author to prove the theorem under much weaker condition for G than a Lie group.

ON THE R_λ-ASSOCIATE OF A LINE

V. G. GROVE

1. Introduction. Popa has shown [4][1] a method of establishing a one-to-one correspondence between lines through a point of a plane net and lines not through the point. Our purpose is to extend this notion to the asymptotic net on a surface. This extension gives a new manner of describing geometrically the R_λ-associate of a line and the R_λ-derived curves as defined [1] by Bell. A simple characterization of the Darboux-Segre pencil is found.

To fix the notation let the coordinates (x^1, x^2, x^3, x^4) of a generic point x on the surface S be functions of the asymptotic parameters u, v; and let these functions be normalized so that they satisfy the differential equations [2]

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + px, \\ x_{vv} &= \gamma x_u + \theta_v x_v + qx, \qquad \theta = \log R. \end{aligned} \tag{1.1}$$

The line joining the points x, x_{uv} is called the R-conjugate line, and that joining x_u, x_v the R-harmonic line of S at x.

Let local coordinates (x_1, x_2, x_3, x_4) referred to the tetrahedron x, x_u, x_v, x_{uv} of a general point X be defined by the formula $X = x_1x + x_2x_u + x_3x_v + x_4x_{uv}$. Let there be given a curve C through x defined by the differential equation $dv - \lambda du = 0$. The equation of the tangent l to C at x is

$$\lambda x_2 - x_3 = 0. \tag{1.2}$$

Moreover let a point z, not in the tangent plane to S at x, be defined by the expression

$$z = x_{uv} - ax_u - bx_v - cx,$$

and denote by h_1 the line xz. The reciprocal h_2 of h_1 with respect to the quadric of Darboux joins the points r, s defined by

$$r = x_u - bx, \qquad s = x_v - ax.$$

2. The R_λ-associate of h_2. Let $y = x(u+\Delta u, v+\Delta v)$ be a point on C near x. The tangent to the asymptotic curve $v = \text{const.}$ at y intersects the plane determined by h_1 and the tangent to $v = \text{const.}$ at x in a point, the limit ρ of whose projection on the tangent plane to S at x

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[1] Numbers in brackets refer to the references cited at the end of the paper.

from the point z as y approaches x along C has coordinates defined by the expression

$$\rho = x_u - \left(b + \frac{\beta}{\lambda}\right)x. \tag{2.1}$$

Similarly a point σ, defined by

$$\sigma = x_v - (a + \gamma\lambda)x, \tag{2.2}$$

may be defined. Let the line determined by ρ, σ be denoted by h.

From the form of (2.1), (2.2), it follows that *h is the R_μ-associate* [1] ($\mu = -\lambda$) of h_2. The equation of h is readily seen to be

$$x_1 + \left(b + \frac{\beta}{\lambda}\right)x_2 + (a + \gamma\lambda)x_3 = 0, \qquad x_4 = 0. \tag{2.3}$$

The envelope of h as l varies in the pencil on x is the conic whose equation is

$$(u_2 - bu_1)(u_3 - au_1) = \beta\gamma u_1^2. \tag{2.4}$$

This conic is tangent to the asymptotic tangents at x at their intersections with the line h_2.

3. **Applications.** The line h is tangent to the conic (2.4) at the point T whose coordinates are

$$(b\gamma\lambda^2 + 2\beta\gamma\lambda + a\beta, \ -\gamma\lambda^2, \ -\beta). \tag{3.1}$$

The equation of the line m determined by x and T is

$$\beta x_2 - \gamma\lambda^2 x_3 = 0. \tag{3.2}$$

The conjugate of m is readily seen to be the R_λ-correspondent [1] *of l*; that is, *it is the conjugate to the tangent to the R_λ-derived curve through x.*

The line m intersects the conic (2.4) *in the points T and T', the tangent at the latter point being the R_λ-associate of h_2.*

Comparing (1.2) and (3.2) we observe that l and m coincide if and only if l is a tangent of Segre, and are conjugate if and only if l is a tangent of Darboux. These theorems are equivalent to Theorems [4.1] and [4.2] of Bell's paper cited in the references.

More generally, one of the cross ratios of the asymptotic tangents and of l and m is $\beta/\gamma\lambda^2$. Hence *all of the tangents to the curves of the Darboux-Segre pencil of conjugate nets are characterized in terms of a constant cross ratio of these four lines.*

We find readily that the locus of the intersection of l and h as l varies in the pencil on x is the rational cubic:

$$x_1x_2x_3 + \beta x_2^3 + bx_2^2x_3 + ax_2x_3^2 + \gamma x_3^3 = 0. \tag{3.3}$$

This cubic is a special case of the cubic (8.7) derived in a similar manner by Bell in the paper [1] cited. *The points of inflexion of the cubic* (3.3) *lie on the Darboux tangents, and on the line* h_2. *We may readily show that the line* h *is tangent to the cubic if and only if* $\lambda^3 = -\beta/\gamma$, *that is, if and only if* l *is a tangent of Darboux. If the line* h_1 *is the R-conjugate line* $(a=b=0)$, *the Hessian of the characteristic cubic defined by us* [3] *is the cubic* (3.3).

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Michigan State College

A NOTE ON CONVERGENCE IN AREA

R. G. HELSEL

Let T_n: $x=x_n(u, v)$, $y=y_n(u, v)$, $z=z_n(u, v)$, $n=0, 1, \cdots$, be a sequence of continuous transformations from the unit square S: $0 \leqq u, v \leqq 1$, into E_3, Euclidean 3-space. For each point p on the unit sphere U: $x^2+y^2+z^2=1$, $E_2(p)$ will denote the plane through the origin which is perpendicular to the radius joining the origin and p; T_p will denote the transformation which projects E_3 perpendicularly onto $E_2(p)$. Then T_pT_n is a sequence of continuous transformations from S into $E_2(p)$. Assume that T_n, $n=1, 2, \cdots$, converges uniformly on S to T_0 and that the Lebesgue area $A(T_n)$ of the F-surface (see [1, II.3.44][1]) determined by the transformation T_n is finite for $n=0, 1, \cdots$. The purpose of the present note is to call attention to the fact that recent results of Helsel [2] and Radó [2, 3] imply the following theorem:

THEOREM. *Under the assumptions stated above, a necessary and sufficient condition for* $A(T_n) \to A(T_0)$ *is that* $A(T_pT_n) \to A(T_pT_0)$ *for every position of the point* p *on* U.

This theorem is the analogue for the Lebesgue area of a new result on convergence in length established by Ayer and Radó [4].

PROOF. The necessity of the condition has been proved by Radó [3]. To establish the sufficiency of the condition, use will be made of the following formula for the Lebesgue area of the F-surface determined by T_n (see [2]):

$$A(T_n) = 2\left[\frac{1}{4\pi}\int\int_U A(T_pT_n)d\sigma\right], \tag{1}$$

$d\sigma$ being the area element on U. In [2] it is shown that the Lebesgue area $A(T_n)$ is equal to twice the integral mean value over U of the lower area $a(T_pT_n)$ of the flat F-surface determined by the transformation T_pT_n; however, the Lebesgue area $A(T_pT_n)$ is equal to the lower area $a(T_pT_n)$ (see [5], [1, V.2.58], and [6]) so (1) follows. The assumption that $A(T_pT_n) \to A(T_pT_0)$ for every point p on U implies, in view of (1), that $A(T_n) \to A(T_0)$ if termwise integration of the sequence $A(T_pT_n)$ is permissible. To show that such is the case, a uniform bound for the functions $A(T_pT_n)$ will be displayed. First ob-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

serve that

$$A(T_pT_n) \leqq A(T_n). \tag{2}$$

Also, by a fundamental result of Cesari [5],

$$A(T_n) \leqq A(T_{p_1}T_n) + A(T_{p_2}T_n) + A(T_{p_3}T_n), \tag{3}$$

where p_1, p_2, p_3 are any three points on U such that the radii joining these points to the origin are mutually perpendicular. Regarding p_1, p_2, p_3 as fixed, the relations $A(T_{p_i}T_n) \to A(T_{p_i}T_0)$, $i=1, 2, 3$, imply the existence of a constant K such that $A(T_{p_i}T_n) \leqq K$ for $n=0, 1, \cdots$ and $i=1, 2, 3$. Hence, from (2) and (3), $A(T_pT_n) \leqq 3K$ for all $p \in U$ and $n=0, 1, \cdots$, which completes the proof of the sufficiency.

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The Ohio State University

ON DISJOINT SETS OF DISTRIBUTION FUNCTIONS

AGNES BERGER

1. **Introduction.** Let us call a function $F(x)$ of the real variable x a distribution function, *d. f.*, if

$$(C_1) \qquad \begin{aligned} &F(x) \text{ is non-decreasing, continuous to the right,} \\ &F(-\infty) = 0, \quad F(+\infty) = 1. \end{aligned}$$

Let further $\{F_i(x)\}$ and $\{G_j(x)\}$ be two sets of d. f.'s with i ranging over the set of values $I = \{i\}$ and j over $J = \{j\}$. We shall assume

$$(C_2) \qquad \begin{aligned} &F_i(x) \not\equiv F_j(x), \quad G_i(x) \not\equiv G_j(x), \\ &F_i(x) \not\equiv G_j(x) \end{aligned} \qquad \text{for all } i \text{ and } j,$$

and attempt to answer the following question proposed by Neyman:

What conditions must be satisfied by sets $\{F_i(x)\}$ and $\{G_j(x)\}$ so that there exists at least one Borel set W such that

$$(A) \qquad \int_W dF_i(x) \neq \int_W dG_j(x) \qquad \text{for all } i, j?$$

This question is of interest in the theory of statistical hypotheses. To indicate this briefly: Let ξ be a random variable of which it is known only that its distribution belongs to the union of the two sets $\{F_i(x)\}$ and $\{G_j(x)\}$. Let H be the hypothesis asserting that the distribution of ξ belongs to $\{F_i(x)\}$. Should there be no Borel set W of the property (A) it would cause difficulty in testing the hypothesis H.

2. **The simplest cases.** To begin with we examine the simplest cases $I=1$, $J=1$ and $I=1$, $J=\{1, 2\}$. It is easy to see that in these cases one can always find an interval solution for (A):

$I=1$, $J=1$. Since by (C_2), there exists an x_0 such that $F_1(x_0) \neq G_1(x_0)$, but by (C_1), $F_1(-\infty) = G_1(-\infty)$, the interval $(-\infty, x_0)$ satisfies (A).

$I=1$, $J=\{1, 2\}$. By (C_2), either of the inequalities $F_1(x) \neq G_j(x)$, $j=1, 2$, has at least one solution. If they have a simultaneous one, say x_0, then $(-\infty, x_0)$ satisfies (A), or else, if whenever $F_1(x_1) \neq G_1(x_1)$, we have $F_1(x_1) = G_2(x_1)$ and whenever $F_1(x_2) \neq G_2(x_2)$, then $F_1(x_2) = G_1(x_2)$ so that (x_1, x_2) satisfies (A).

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3. Discontinuous d. f.'s. So far no restrictions have been imposed upon the d. f.'s. Now we show that as long as the d. f.'s are allowed to have jumps, (A) does not always admit a solution when $I=1$, $J=\{1, 2, 3\}$. Namely we shall prove the following theorem:

THEOREM I. *Whatever be the distribution function $F(x)$, as long as it has at least two discontinuities, it is always possible, and in a continuum of ways, to determine three different distribution functions $G_j, j=1, 2, 3$, each different from $F(x)$ and such that for every Borel set W*

$$\int_W dF = \int_W dG_j \qquad \text{for some } j.$$

PROOF. Let $a_1<a_2$ be two discontinuity points of $F(x)$ and $a_3>a_2$ an arbitrary point on the x-axis. Let the set of three points a_1, a_2, a_3 be denoted by σ. Let the saltus of $F(x)$ at a_1, a_2, a_3 be respectively p_1, p_2, p_3, where $p_1>0$, $p_2>0$, but $p_3\geqq 0$ and $\sum_{i=1}^{3} p_i=s\leqq 1$. Then every Borel set W can be represented as a sum $W'+W''$, where $W'=\sigma W$ and $W''=W-W'$ and

$$\int_W dF = \int_{W'} dF + \int_{W''} dF.$$

Let $a_0=-\infty$, $a_4=+\infty$. Let $G_j(x), j=1, 2, 3$, be three d. f.'s satisfying the relations $G_j(x)-G_j(a_k)=F(x)-F(a_k)$ for $a_k\leqq x<a_{k+1}$ for $k=0, 1, 2, 3$. Then, for every W'' disjoint with σ

$$(*) \qquad \int_{W''} dG_j(x) = \int_{W''} dF(x) \qquad \text{for every } j.$$

In order to complete the definition of the functions $G_j(x)$ it remains to specify their saltus at the points a_1, a_2, a_3. Thus let us denote the saltus of G_j at a_i by q_{ji}, $j, i=1, 2, 3$, and put

$$q_{ji} = p_i, \qquad j = 1, 2, 3,$$

while we choose arbitrarily any q_{ji} such that

$$0 \leqq q_{ji} \neq p_i \qquad \text{for } i \neq j;\ i, j = 1, 2, 3,$$

and satisfying $\sum_{i=1}^{3} q_{ji}=s$, $j=1, 2, 3$. Then $G_j\not\equiv F$, $j=1, 2, 3$, but for $W'=a_i$, $i=1, 2, 3$,

$$\int_{W'} dF = p_j = \int_{W'} dG_j \qquad \text{for some } j.$$

For $W'=\sigma-a_i$, $i=1, 2, 3$,

$$\int_{W'} dF = s - p_j = \int_{W'} dG_j \qquad \text{for some } j,$$

and since

$$\int_{e} dF = \int_{e} dG_j \qquad \text{for all } j,$$

we have by (*), for arbitrary W,

$$\int_{W} dF = \int_{W} dG_j \qquad \text{for some } j.$$

Using Theorem I, we can show now that also in the case $I=\{1, 2\}$, $J=\{1, 2\}$, there exist sets of discontinuous d. f.'s for which (A) can not be satisfied. Namely let F, G_1, G_2, G_3 be a set of d. f.'s satisfying Theorem I and suppose further that $p_2<q_{32}<2p_2$. (This is obviously allowed by the construction.) Then $p_i+q_{2i}-q_{3i}\geqq 0$ for $i=1, 2, 3$, so that the function $F(x)+G_2(x)-G_3(x)$ is a d. f.

Let now

$$F_1^* = F, \qquad F_2^* = F + G_2 - G_3, \qquad G_1^* = G_1, \qquad G_2^* = G_3.$$

The sets $\{F_i^*\}$, $\{G_j^*\}$, $i=1, 2$, $j=1, 2$, obviously satisfy (C_1) and (C_2). Let $F-G_j=D_j$, $j=1, 2, 3$, and $F_i^*-G_j^*=D_{ij}^*$, $i=1, 2$, $j=1, 2$. Then

$$D_{11}^* = D_1, \qquad D_{12}^* = D_2, \qquad D_{21}^* = F + G_2 - G_3 - G_1, \qquad D_{22}^* = D_3,$$

so that the set $\{D_{ij}^*\}$, $i=1, 2, j=1, 2$, contains the set $\{D_j\}$, $j=1, 2, 3$. Since, by assumption, whatever be W, $\int_W dF-\int_W dG_j=\int_W d(F-G_j)=\int_W dD_j=0$ for at least one of the values $j=1, 2, 3$; also,

$$\int_{W} dD_{ij}^* = 0 \qquad \text{for at least one pair } i, j.$$

Therefore we can state the following theorem:

THEOREM II. *Whenever the number of distribution functions in the set $\{F_i\}$ multiplied by the number of distribution functions in the set $\{G_j\}$ is not less than 3 and the distribution functions are allowed to have jumps, it is not true that to every such pair of disjoint sets of distribution functions a Borel set W exists for which*

$$\int_{W} dF_i - \int_{W} dG_j \neq 0 \qquad \textit{for all } i, j.$$

4. Continuous distribution functions. On the other hand we shall show that if the d. f.'s are continuous, the following theorem holds:

THEOREM III. *Let* $\{F_i\}$ *and* $\{G_j\}$ *be two denumerable sequences of distribution functions, each satisfying* (C_1) *and* (C_2) *and each continuous in* x. *Then there always exists a Borel set* W *of arbitrarily small measure and such that*

$$\int_W dF_i - \int_W dG_j \neq 0 \qquad \textit{for all } i, j.$$

In the proof we shall make repeated use of the terms "point of change of a function" and "interval of change of a function," defined as follows:

Let f be any function defined on a closed interval $[X, Y]$ and such that

$$f(X) \neq f(Y).$$

Let Z be the least upper bound of those points z of the interval $[X, Y]$ for which $f(z) \neq f(Y)$. Z will be called the point of change of f in $[X, Y]$. Obviously Z has the property that given any $\delta > 0$, there exists a point z for which

$$|z - Z| < \delta$$

and

$$f(z) \neq f(Z).$$

We shall say that z and Z determine an interval of change of f in the δ-neighborhood of Z.

PROOF OF THEOREM III. Let $F_i(x) - G_j(x) = D_{ij}(x)$. Since

$$\int_W dF_i - \int_W dG_j = \int_W d(F_i - G_j),$$

we have to find a solution for

$$\text{(A')} \qquad \int_W dD_{ij}(x) \neq 0 \qquad \text{for all } i, j.$$

Since the sequences $\{F_i\}$ and $\{G_j\}$ are enumerable, so is the set $\{D_{ij}\}$ and its elements may be enumerated in a single indexed sequence

$$D_1(x), D_2(x), \cdots, D_i(x), \cdots.$$

Thus (A′) becomes

$$\text{(A}''\text{)} \qquad \int_W dD_i(x) \neq 0 \qquad \text{for all } i$$

or, introducing the notation

$$\text{(A}'''\text{)} \qquad \begin{aligned} &\int_W dD_i(x) = V_i(W), \\ &V_i(W) \neq 0 \qquad \text{for all } i. \end{aligned}$$

By (C_1), (C_2), the functions $D_i(x)$ satisfy the following conditions:

$$\text{(C}_{12}\text{)} \qquad \begin{aligned} &D_i(-\infty) = D_i(+\infty) = 0, \\ &D_i(X_i) \neq 0 \qquad \text{for at least one finite point } X_i. \end{aligned}$$

But since $D_i(x)$ is continuous, for each $D_i(x)$ there exists at least one other finite point Y_i such that $D_i(Y_i) \neq 0$ and

$$\text{(C}_3\text{)} \qquad D_i(X_i) \neq D_i(Y_i).$$

Therefore we can assign to every $D_i(x)$ a point of change Z_i and to every given δ an interval of change in the δ-neighborhood of Z_i.

Let now $\epsilon > 0$ be given in advance. First we shall show that it is possible to construct a sequence of Borel sets $W_1, W_2, \cdots, W_r, \cdots$ with the following properties:

(i) W_r consists of a finite number of disjoint intervals;

(ii) The measure of W_r is smaller than $\epsilon/2^r$;

(iii) $V_1(W_1) \neq 0$;

(iv) If $V_i(W_1) \neq 0$ for $i \leqq i_1$ but $V_{i_1+1}(W_1) = 0$, then there exists an $i_2 > i_1$ such that $V_i(W_2) \neq 0$ for $i \leqq i_2$ and, in general, if $V_i(W_r) \neq 0$ for $i \leqq i_r$, but $V_{i_r+1}(W_r) = 0$, then $V_i(W_{r+1}) \neq 0$ for $i \leqq i_{r+1} > i_r$.

If in the process of construction we arrive at a W_r such that $V_i(W_r) \neq 0$ for all i, the theorem is proved. If, however, we arrive at an infinite sequence, we shall show that this sequence has a non-empty limit set W which satisfies the requirements of the theorem.

The construction of the sequence W_r is done by induction: Let Z_1 be a point of change of D_1 and W_1 an interval of change of D_1 in the $\epsilon/2$-neighborhood of Z_1. Then (iii) holds. Assume now that we succeeded in finding r terms of the sequence, $W_1, W_2, \cdots, W_s, W_{s+1}, \cdots, W_r$ such as requested. Then we know that

$$\begin{aligned} &\text{(1)} \qquad V_i(W_s) \neq 0, \quad i \leqq i_s, \\ &\text{(2)} \qquad V_{i_s+1}(W_s) = 0. \end{aligned} \qquad \text{for } s = 1, 2, \cdots, r,$$

Let us for brevity put $i_r + 1 = \nu$ and let Z_ν be a point of change of D_ν.

Since by assumption W_r consists of a finite number of disjoint intervals, we can and should choose a δ_r such that $0<\delta_r<\epsilon/2^r$ and so small that the δ_r-neighborhood of Z_r, and therefore every interval of change in it, should not contain any boundary point of W_r except perhaps Z_r.

By (1) for every $s \leqq r$, there exists an $\eta_s>0$ such that

$$|V_i(W_s)| > \eta_s, \qquad i \leqq i_s,\ s \leqq r. \tag{3}$$

Let now $\sum_1^\infty \alpha_i=\alpha<1$, $\alpha_i>\alpha_{i+1}>0$, be any given series and $[A_r, B_r]$ any closed interval $A_r<Z_r<B_r$. Since $D_1, D_2, \cdots, D_{i_r}$ are continuous in $[A_r, B_r]$, δ_r can and should be further adjusted to be so small, that for every subinterval $I(\delta_r)$ of length smaller than δ_r in $[A_r, B_r]$

$$|V_i[I(\delta_r)]| < \eta_s\alpha_r, \qquad s \leqq r,\ i \leqq i_r, \tag{4}$$

should hold.

Having δ_r thus fixed, let Δ_r be an interval of change of D_r in the δ_r-neighborhood of Z_r, so that

$$|\Delta_r| < \delta_r < \epsilon/2^r \tag{5}$$

and

$$|V_s(\Delta_r)| = |V_{s+1}(\Delta_r)| > 0. \tag{6}$$

Let now

$$W_{r+1} = W_r + (-1)^{p_r}\Delta_r,$$

where $p_r=2$ if Δ_r contains no inner points of W_r, and $p_r=1$ if Δ_r contains inner but no outer points of W_r. Then by (3), (4), and (5)

$$|V_i(W_{r+1})| \geqq |V_i(W_r)| - |V_i(\Delta_r)| > \eta_r - \eta_r\alpha_r > 0 \qquad \text{for } i \leqq i_r$$

and by (2) and (6)

$$|V_{i+1}(W_{r+1})| = |V_{i+1}(\Delta_r)| > 0.$$

Let us now put

$$W = W_1 + \sum_1^\infty (-1)^{p_r}\Delta_r.$$

By (5), W is not empty and its measure is less than ϵ. By (4) we have

$$|V_i(\Delta_r)| < |V_i(W_s)|\,\alpha_r, \quad i \leqq i_s,\ s = 1, 2, \cdots, r;\ r = 1, 2, \cdots. \tag{7}$$

Let i be a fixed arbitrary index. Find in the sequence $i_1, i_2, \cdots$ the indices i_{k-1}, i_k for which

$$i_{k-1} < i \leqq i_k.$$

By (7)

$$| V_i(W) | \geqq | V_i(W_k) | - \sum_{j=0}^{\infty} | V_i(\Delta_{k+j}) |,$$

while for any j

$$| V_i(\Delta_{k+j}) | < | V_i(W_k) | \alpha_{k+j},$$

thus

$$\sum_{j=0}^{\infty} | V_i(\Delta_{k+j}) | < | V_i(W_k) | \sum_{0}^{\infty} \alpha_{k+j}$$

and

$$| V_i(W) | > | V_i(W_k) | \left(1 - \sum_{0}^{\infty} \alpha_{k+j}\right) \geqq 0,$$

$$| V_i(W) | > 0,$$

Q.E.D.

Further results concerning the case when I and J range over a continuum of indices are discussed in a joint paper with Wald in one of the recent issues of the Annals of Mathematical Statistics.

While preparing this paper I had opportunities to discuss it with Professor Neyman and I wish to acknowledge my indebtedness for his generous advice and many helpful suggestions.

NEW YORK CITY

A DERIVATION OF ZORAWSKI'S CRITERION FOR PERMANENT VECTOR-LINES[1]

R. PRIM AND C. TRUESDELL

The classical hydrodynamic theorems of Helmholtz[2] state that in any motion of an inviscid, incompressible fluid subject to conservative extraneous force the vortex-tubes are material tubes whose strength is constant in space and time. They were extended to barotropic motions of compressible fluids by Kelvin[3] and Nanson.[4] Underlying these theorems of hydrodynamics are three purely kinematical questions:

1. What is a necessary and sufficient condition that the strength of the vector-tubes of a continuously differentiable vector field $\mathbf{c}$ be the same at all cross-sections?

2. Given a continuously differentiable velocity field $\mathbf{v}(\mathbf{r}, t)$, what is a necessary and sufficient condition that the strength of the vector-tubes of a second continuously differentiable field $\mathbf{c}(\mathbf{r}, t)$ remain constant throughout the motion?

3. Given a continuously differentiable velocity field $\mathbf{v}(\mathbf{r}, t)$, what is a necessary and sufficient condition that the vector-tubes of a second continuously differentiable field $\mathbf{c}(\mathbf{r}, t)$ be material tubes?

The answer to the first question was given by Kelvin,[5] prior to the work of Helmholtz, the required condition being

$$\operatorname{div} \mathbf{c} = 0. \tag{1}$$

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[1] This investigation was carried out under contract No. 53-47, *Mechanics of deformable continua*, from the Office of Naval Research to the Naval Ordnance Laboratory, while the authors were employed by the latter institution.

[2] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, J. Reine Angew. Math. vol. 55 (1858) pp. 25–55; reprinted in *Wissenschaftliche Abhandlungen von Hermann Helmholtz*, vol. 1, Leipzig 1882, pp. 101–134. See §2.

[3] W. Thomson, *On vortex motion*, Transactions of the Royal Society of Edinburgh vol. 25 (1869) pp. 217–260; reprinted *Mathematical and physical papers of Lord Kelvin*, vol. 4, Cambridge, 1910, pp. 13–66. See §§59–60(r).

[4] E. J. Nanson, *Note on hydrodynamics*, Messenger of Mathematics vol. 3 (1874) pp. 120–121. The derivations of Helmholtz and Nanson are reproduced and criticised in H. Lamb, *Hydrodynamics*, 6th ed., Cambridge, 1932, pp. 203–206.

[5] W. Thomson, *A mathematical theory of magnetism*, Philosophical Transactions of the Royal Society of London vol. 141 (1851) pp. 243–245; reprinted *Papers on electrostatics and magnetism*, §§432–523. See §74 of former, §513 of latter.

The answer to the second question was given by Zorawski,[6] the required condition being

$$\frac{D\mathbf{c}}{Dt} - \mathbf{c}\cdot\mathbf{grad}\ \mathbf{v} + \mathbf{c}\ \mathrm{div}\ \mathbf{v} = 0, \tag{2}$$

where D/Dt is the symbol of material differentiation. This result is really an immediate consequence of a formula given earlier by Lamb.[7] The answer to the third question was also given by Zorawski,[8] the required condition being

$$\mathbf{c} \times \left[\frac{D\mathbf{c}}{Dt} - \mathbf{c}\cdot\mathbf{grad}\ \mathbf{v} + \mathbf{c}\ \mathrm{div}\ \mathbf{v}\right] = 0, \tag{3}$$

which was implicit in the infinitesimal analysis of Helmholtz and Nanson. The purpose of the present note is to give a simple vectorial derivation of the criterion (3).

Let a single material line be given by $\mathbf{r}=\mathbf{r}(\theta, t)$, where θ is a parameter along the curve and t is the time; the same coordinate is associated with each material point at all times, so that θ and t are independent variables. Then

$$\begin{aligned}\frac{D}{Dt}\left(\frac{\partial \mathbf{r}}{\partial\theta}\times\mathbf{c}\right) &= \frac{\partial \mathbf{v}}{\partial\theta}\times\mathbf{c} + \frac{\partial \mathbf{r}}{\partial\theta}\times\frac{D\mathbf{c}}{Dt},\\ &= \left(\frac{\partial \mathbf{r}}{\partial\theta}\cdot\mathbf{grad}\ \mathbf{v}\right)\times\mathbf{c} + \frac{\partial \mathbf{r}}{\partial\theta}\times\frac{D\mathbf{c}}{Dt}.\end{aligned} \tag{4}$$

Now if at the instant $t=0$ the material curve in question is a vector-line of $\mathbf{c}$, we shall have

$$\frac{\partial \mathbf{r}}{\partial\theta}\times\mathbf{c} = 0, \quad \text{or} \quad \frac{\partial \mathbf{r}}{\partial\theta} = \lambda\mathbf{c}, \tag{5}$$

where λ is a scalar function. Then (4) becomes

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[6] K. Zorawski, *Ueber die Erhaltung der Wirbelbewegung*, Bulletin de l'Académie des Sciences de Cracovie, Comptes Rendus (1900) pp. 335–342. Preferable proofs are given by G. Jaumann, *Die Grundlagen der Bewegungslehre von einem modernen Standpunkte aus*, Leipzig, 1905, see §383; J. Spielrein, *Lehrbuch der Vektorrechnung nach den Bedürfnissen in der technischen Mechanik und Elektrizitätslehre*, Stuttgart, 1916, see §29; E. Lohr, *Vektor- und Dyadenrechnung für Physiker und Techniker*, Berlin, 1939, Part III, chap. 5, §f; C. A. Truesdell and R. C. Prim, *Zorawski's kinematic theorems*, Naval Ordnance Laboratory Memorandum, 9354, 1947.

[7] H. Lamb, *Note on a theorem in hydrodynamics*, Messenger of Mathematics vol. 7 (1877–1878) pp. 41–42.

[8] Loc. cit.

$$\frac{D}{Dt}\left(\frac{\partial \boldsymbol{r}}{\partial \theta} \times \mathbf{c}\right) = \lambda \mathbf{c} \times \left[\frac{D\mathbf{c}}{Dt} - \mathbf{c} \cdot \operatorname{grad} \boldsymbol{v}\right]. \tag{6}$$

A necessary and sufficient condition that the material line remain a vector-line of $\mathbf{c}$ is that (5) hold at all times. Since $\mathbf{c} \times \mathbf{c} = 0$, from (6) we may then deduce the necessity of (3). Conversely, if (3) be satisfied, then $\partial \boldsymbol{r}/\partial\theta \times \mathbf{c}$ at each point on the material line is a quantity initially zero whose time derivative is always zero, and hence itself remains zero, so that (3) is also sufficient.

Since (3) is a consequence of (2), but (2) is not generally a consequence of (3), in order for the vector-tubes of $\mathbf{c}$ to be of strength constant in time it is necessary, but not sufficient, for the vector-tubes to be material tubes. The criterion (1) has not been used in the deduction of (2) or (3); thus even if the vector-tubes are of strength constant in time at each material cross-section they are not generally of equal strength at all cross-sections.

If $\mathbf{c} = \operatorname{curl} \boldsymbol{v}$, (1) is satisfied and (2) and (3) become respectively[9]

$$\operatorname{curl} \boldsymbol{a} = 0, \tag{7}$$

$$\operatorname{curl} \boldsymbol{v} \times \operatorname{curl} \boldsymbol{a} = 0, \tag{8}$$

where $\boldsymbol{a}$ is the acceleration. It is easy to show that (7) is a necessary and sufficient condition that the circulation around an arbitrary closed material curve be constant throughout the motion. From (8) it follows that the constancy of circulation is sufficient but not necessary in order to ensure the permanence of the vector-tubes.

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[9] The formula (8) and an incorrect substitute for (7) were stated without proof by M. Levy, *L'hydrodynamique moderne et l'hypothèse des actions à distance*, Révue Générale des Sciences Pures et Appliquées vol. 1 (1890) pp. 721–728. Proofs are given by H. Poincaré, *Théorie des Tourbillons*, Paris, 1893, §§5–6, pp. 150–151. See also Jaumann, op. cit. §386; E. Vessiot, *Sur les transformations infinitésimales et la cinématique des milieux continus*, Bull. Sci. Math. (2) vol. 35, part 1, pp. 233–244, §4; P. Appell, *Traité de mécanique rationnelle*, vol. 3, 3d ed., Paris, 1921, chap. 25, ex. 5.

SPACES WITH A COMPACT LIE GROUP OF TRANSFORMATIONS

A. M. GLEASON

Introduction. A topological group $\mathfrak{G}$ is said to act on a topological space $\mathfrak{R}$ if the elements of $\mathfrak{G}$ are homeomorphisms of $\mathfrak{R}$ onto itself and if the mapping $(\sigma, p) \to \sigma(p)$ of $\mathfrak{G} \times \mathfrak{R}$ onto $\mathfrak{R}$ is continuous. Familiar examples include the rotation group acting on the Cartesian plane and the Euclidean group acting on Euclidean space. The set $\mathfrak{G}(p)$ (that is, the set of all $\sigma(p)$ where $\sigma \in \mathfrak{G}$) is called the orbit of p. If p and q are two points of $\mathfrak{R}$, then $\mathfrak{G}(p)$ and $\mathfrak{G}(q)$ are either identical or disjoint, hence $\mathfrak{R}$ is partitioned by the orbits. The topological structure of the partition becomes an interesting question. In the case of the rotations of the Cartesian plane we find that, excising the singularity at the origin, the remainder of space is fibered as a direct product. A similar result is easily established for a compact Lie group acting analytically on an analytic manifold. In this paper we make use of Haar measure to extend this result to the case of a compact Lie group acting on any completely regular space. The exact theorem is given in §3. §§1 and 2 are preliminaries, while in §4 we apply the main result to the study of the structure of topological groups. These applications form the principal motivation of the entire study,[1] and the author hopes to develop them in greater detail in a subsequent paper.

1. An extension theorem. In this section we shall prove an extension theorem which is an elementary generalization of the well known theorem of Tietze. For the sake of completeness we include the proof although it proceeds along standard lines.

We recall that a topological space $\mathfrak{R}$ is said to be completely regular if it is a Hausdorff space and if, for any closed set F and point $p \notin F$, there exists a continuous, real-valued function f defined on $\mathfrak{R}$ such that $f(p) = 1$ and $f(q) = 0$ if $q \in F$. If we can choose such a function, we can choose another which is also non-negative; for example, let $h(q) = \max(0, f(q))$.

1.1 LEMMA. *Let F be a closed subset of a completely regular space $\mathfrak{R}$.*

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[1] The author began this study thinking only in terms of coset decompositions of topological groups. He is indebted to Professor Deane Montgomery for suggesting Theorem 3.4.

Let C be a compact subset of $\mathfrak{R}$ disjoint from F. Let λ be a positive real number. Then there exists a continuous, real-valued function g defined on $\mathfrak{R}$ such that:

(1) $0 \leqq g(q) \leqq \lambda$ *for all* $q \in \mathfrak{R}$.
(2) $g(q) = 0$ *if* $q \in F$.
(3) $g(q) = \lambda$ *if* $q \in C$.

PROOF. For each $p \in C$ we choose a continuous non-negative function f_p such that $f_p(p) = 1$ and $f_p(q) = 0$ if $q \in F$. Let $V(p) = \hat{q}(f_p(q) > 1/2)$. Since f_p is continuous, $V(p)$ is an open neighborhood of p; hence $C \subset \bigcup_{p \in C} V(p)$. C, being compact, is covered by a finite subcollection of the $V(p)$, say $C \subset \bigcup_{i=1}^{n} V(p_i)$. Define $g(q) = \min(\lambda, 2\lambda \sum_{i=1}^{n} f_{p_i}(q))$. The function g is easily verified to have the required properties.

1.2 DEFINITION. A real-valued function f is said to vanish at infinity if, for any $\epsilon > 0$, there is a compact set C such that $|f(p)| < \epsilon$ for all $p \notin C$.

The functions which vanish at infinity form a linear class.

1.3 LEMMA. *Let D be a closed subset of a completely regular space $\mathfrak{R}$. Let f be a continuous, non-negative function defined on D and vanishing at infinity such that $f(p) \leqq k$ for all $p \in D$. Then there exists a function g defined on $\mathfrak{R}$ such that;*

(1) $0 \leqq g(p) \leqq k/3$ *for all* $p \in \mathfrak{R}$.
(2) $0 \leqq f(p) - g(p) \leqq 2k/3$ *for all* $p \in D$.
(3) g *vanishes at infinity on* D.

PROOF. We assume that $k > 0$, otherwise the lemma is trivial. Let $C = \hat{p}\,(p \in D, f(p) \geqq 2k/3)$ and $F = \hat{p}\,(p \in D, f(p) \leqq k/3)$. These sets are closed. Since f vanishes at infinity the sets C and $\mathrm{Cl}(D - F)$ are compact. We apply Lemma 1.1 to the sets C and F, choosing $\lambda = k/3$. We obtain a function g which satisfies our requirements. For g vanishes on all of D except the compact set $\mathrm{Cl}(D - F)$, and the inequalities of (2) follow from

$$\begin{aligned} 2k/3 \leqq f(p) \leqq k, &\qquad g(p) = k/3 &&\text{for } p \in C, \\ k/3 \leqq f(p) \leqq 2k/3, &\qquad 0 \leqq g(p) \leqq k/3 &&\text{for } p \in D - C - F, \\ 0 \leqq f(p) \leqq k/3, &\qquad g(p) = 0 &&\text{for } p \in F. \end{aligned}$$

1.4 THEOREM. *Let D be a closed subset of a completely regular space $\mathfrak{R}$. Let f be a continuous, real-valued function defined on D and vanishing at infinity such that $|f(p)| \leqq k$ for all $p \in D$. Then there exists a continuous, real-valued function h defined on $\mathfrak{R}$ such that $|h(p)| \leqq k$ for all $p \in \mathfrak{R}$ and $h(p) = f(p)$ for all $p \in D$.*

PROOF. We assume to begin with that f is non-negative. We apply the preceding lemma to f and obtain a function g_1 defined on $\mathcal{R}$ and vanishing at infinity on D such that $0 \leqq g_1(p) \leqq k/3$ for all $p \in \mathcal{R}$ and $0 \leqq f(p) - g_1(p) \leqq 2k/3$ for all $p \in D$. Since the function $f - g_1$ is defined and vanishes at infinity on D we can apply the lemma again and obtain a function g_2 defined on $\mathcal{R}$ and vanishing at infinity on D such that $0 \leqq g_2(p) \leqq 2k/3^2$ for all $p \in \mathcal{R}$ and $0 \leqq f(p) - g_1(p) - g_2(p) \leqq 2^2k/3^2$ for all $p \in D$. Applying the lemma repeatedly, we obtain a sequence of functions $\{g_n\}$ each defined on $\mathcal{R}$ and vanishing at infinity on D such that:

(1) $0 \leqq g_n(p) \leqq 2^{n-1}k/3^n$ for all $p \in \mathcal{R}$.

(2) $0 \leqq f(p) - \sum_{i=1}^{n} g_i(p) \leqq 2^n k/3^n$ for all $p \in D$.

Put $h(p) = \sum_{i=1}^{\infty} g_i(p)$. From (1) it follows that the series is uniformly convergent and, therefore, that h is continuous; moreover, $h(p) \leqq k$ for all $p \in \mathcal{R}$. It follows from (2) that $h(p) = f(p)$ for all $p \in D$. This completes the proof in the case that f is non-negative. We note that in this case the extended function turned out to be non-negative.

In the general case we put $f_1(p) = \max(0, f(p))$ and $f_2(p) = \max(0, -f(p))$. Then f_1 and f_2 are non-negative functions which vanish at infinity on D with the same bound as f; furthermore, $f = f_1 - f_2$. Let h_1 and h_2 be non-negative extensions of f_1 and f_2 respectively, obtained as above. Then $h = h_1 - h_2$ is the required extension of f.

In what follows we shall not need the full force of this theorem, but only the fact that we can extend any function from a compact set to the whole of $\mathcal{R}$.

2. Topological results. In this section we study the action of a topological group on a topological space without any assumptions of an analytic character. Theorem 2.3 is a useful criterion for determining if the orbit partition is indeed a direct product fibering.

2.1 LEMMA. *Let $\mathfrak{G}$ be a topological group which acts on a topological space $\mathcal{R}$. If C is a compact subset of $\mathfrak{G}$ and F is a closed subset of $\mathcal{R}$, then $C(F)$ is closed.*

PROOF. Suppose $p \notin C(F)$. We must show that there is a neighborhood V of p such that $V \subset \mathcal{R} - C(F)$. For each $\sigma \in C$, $\sigma^{-1}(p) \in \mathcal{R} - F$. Because the latter set is open, we can choose neighborhoods U_σ of σ^{-1} and V_σ of p such that $U_\sigma(V_\sigma) \subset \mathcal{R} - F$. C^{-1}, being compact, is covered by the union of a finite subcollection of the U_σ, say $C^{-1} \subset \bigcup_{i=1}^{n} U_{\sigma_i}$. Put $V = \bigcap_{i=1}^{n} V_{\sigma_i}$. V is a neighborhood of p. Now $U_{\sigma_i}(V) \subset U_{\sigma_i}(V_{\sigma_i}) \subset \mathcal{R} - F$; hence $C^{-1}(V) \subset \bigcup_{i=1}^{n} U_{\sigma_i}(V) \subset \mathcal{R} - F$, from which it follows that $V \subset \mathcal{R} - C(F)$.

2.2 LEMMA. *Let $\mathfrak{G}$ be a topological group which acts on a topological space $\mathfrak{R}$. If C is a compact subset of $\mathfrak{G}$ and E is any subset of $\mathfrak{R}$, then* $C(\mathrm{Cl}(E)) = \mathrm{Cl}(C(E))$.

PROOF. For any element $\sigma \in C$ we have $\sigma(E) \subset C(E)$, hence $\mathrm{Cl}(\sigma(E)) \subset \mathrm{Cl}(C(E))$. Since σ is a homeomorphism, $\mathrm{Cl}(\sigma(E)) = \sigma(\mathrm{Cl}(E))$, hence $C(\mathrm{Cl}(E)) = \bigcup_{\sigma \in C} \sigma(\mathrm{Cl}(E)) \subset \mathrm{Cl}(C(E))$. On the other hand, $C(E) \subset C(\mathrm{Cl}(E))$ and, since the latter set is closed by the preceding lemma, $\mathrm{Cl}(C(E)) \subset C(\mathrm{Cl}(E))$.

In what follows we shall denote by $\mathfrak{G}_p$ the set of all elements of $\mathfrak{G}$ which leave the point p fixed. Evidently $\mathfrak{G}_p$ is a closed subgroup of $\mathfrak{G}$.

2.3 THEOREM. *Let $\mathfrak{G}$ be a compact topological group which acts on a topological space $\mathfrak{R}$. Let $\mathfrak{K}$ be a closed subgroup of $\mathfrak{G}$. Let C be a closed subset of $\mathfrak{R}$ such that if $p \in C$ then $\mathfrak{G}_p = \mathfrak{K}$ and $\mathfrak{G}(p) \cap C = \{p\}$. Then the mapping $(\sigma\mathfrak{K}, p) \to \sigma(p)$ of $(\mathfrak{G}/\mathfrak{K}) \times C$ onto $\mathfrak{G}(C)$ is a homeomorphism.*

PROOF. It is clear that the mapping $(\sigma\mathfrak{K}, p) \to \sigma(p)$ where $p \in C$ is well-defined. The continuity of this mapping follows from the continuity of $(\sigma, p) \to \sigma(p)$ and the definition of the topology in $\mathfrak{G}/\mathfrak{K}$. Suppose that $\sigma(p) = \tau(q)$ where $p, q \in C$. Then $\tau^{-1}\sigma(p) = q$ and it follows that $p = q$ and $\tau^{-1}\sigma \in \mathfrak{K}$; τ and σ determine the same left coset, and hence the mapping is one-to-one. It remains to show that it carries open sets into open sets. For this purpose we may confine ourselves to open sets of the type $U \times V$ where U and V are open sets of $\mathfrak{G}/\mathfrak{K}$ and C respectively. Since the mapping is one-to-one we need only show that the complements of such sets are carried into closed sets. The complement of $U \times V$ is $(((\mathfrak{G}/\mathfrak{K}) - U) \times C) \cup ((\mathfrak{G}/\mathfrak{K}) \times (C - V))$, and its image is $(\mathfrak{G} - U_1)(C) \cup \mathfrak{G}(C - V)$ where U_1 is the complete inverse image of U under the natural map of $\mathfrak{G}$ onto $\mathfrak{G}/\mathfrak{K}$. By Lemma 2.1 this image is closed and the theorem is proved.

3. **Principal results.** We now consider the case of a compact Lie group acting on a completely regular space. Beginning with the simplest case in which we are aided by the existence of analytic coordinates, we show that it is possible to construct local "cross-sections"; that is, closed sets which meet the orbit of each nearby point just once. The importance of these sets is reflected in Theorem 2.3. The principal weapon is the use of Haar measure to produce an "orbit-faithful" representation of the group acting on the space.

3.1 LEMMA. *Let $\mathfrak{G}$ be a closed subgroup of a Lie group $\mathfrak{M}$. Then there exists a compact neighborhood N of the identity in $\mathfrak{M}$ and a closed subset C of N such that every right coset of $\mathfrak{G}$ which intersects N has exactly one point in common with C.*

Since this result is standard in the theory of Lie groups, the proof will not be given here. A proof can be found in Chevalley, *Theory of Lie groups*: I, Princeton University Press, 1946, pp. 109–110.[2]

3.2 LEMMA. *Let $\mathfrak{G}$ be a compact group of $n\times n$ matrices. We assume that $\mathfrak{G}$ acts on the Cartesian space of all $n\times n$ matrices by matrix multiplication. Then there exists a closed neighborhood N of the identity matrix and a closed subset C of N each that the orbit of every point of N has exactly one point in common with C. We can choose N so that it contains only nonsingular matrices.*

PROOF. The Cartesian space $\mathfrak{S}$ of all $n\times n$ matrices contains the group $\mathfrak{M}$ of all nonsingular $n\times n$ matrices as an open subspace. $\mathfrak{G}$ is a subgroup of $\mathfrak{M}$, and the orbits under $\mathfrak{G}$ of points in $\mathfrak{M}$ are just the right cosets of $\mathfrak{G}$ in $\mathfrak{M}$. Being compact, $\mathfrak{G}$ is a closed subgroup of $\mathfrak{M}$, which is, of course, a Lie group. We choose the sets N and C as in Lemma 3.1. Then N is a closed neighborhood of the identity matrix in $\mathfrak{S}$ which contains only nonsingular matrices and C has exactly one point in common with the orbit of each point of N.

3.3 THEOREM. *Let $\mathfrak{G}$ be a compact Lie group which acts on a completely regular space $\mathfrak{R}$. Let p be a point of $\mathfrak{R}$ such that $\sigma(p)\neq p$ unless σ is the identity. Then there exists a closed neighborhood N of p and a closed subset C of N such that the orbit of every point of N has exactly one point in common with C.*

PROOF. Let H be a faithful representation of $\mathfrak{G}$ as a group of $n\times n$ matrices.[3] We shall define a function J on $\mathfrak{R}$ with values in the space of all $n\times n$ matrices, that is to say, n^2-dimensional Cartesian space. On the orbit of p let $J(\sigma(p))=H(\sigma)$ (the condition that $\sigma(p)\neq p$ unless σ is the identity insures against inconsistency) and let J be extended to the rest of $\mathfrak{R}$ in any continuous fashion. This extension is possible by virtue of Theorem 1.4 and the fact that $\mathfrak{G}(p)$, as the image of a compact set, is compact. Define a new matrix valued function K by setting

$$K(q)=\int_{\mathfrak{G}} H(\tau^{-1})J(\tau(q))d\tau,$$

where the integration is with respect to the Haar measure of $\mathfrak{G}$. The

[2] Our statement of the lemma does not agree with Chevalley's Proposition 1, but the proof of the latter contains a complete proof of Lemma 3.1.

[3] The Peter-Weyl theorem assures us of the existence of a faithful matrix representation (cf. Pontrjagin, *Topological groups*, Princeton University Press, 1939, pp. 87–125).

hypothesis assures that $\tau(q)$ is a continuous function of τ and q together, whence it follows easily that K is continuous. The important relation $K(\sigma(q))=H(\sigma)K(q)$ follows from the invariant property of the integration, for

$$K(\sigma(q)) = \int_{\mathfrak{G}} H(\tau^{-1})J(\tau\sigma(q))d\tau = \int_{\mathfrak{G}} H(\sigma\rho^{-1})J(\rho(q))d\rho$$

$$= H(\sigma)\int_{\mathfrak{G}} H(\rho^{-1})J(\rho(q))d\rho = H(\sigma)K(q).$$

Moreover, $K(p)=\int_{\mathfrak{G}}H(\tau^{-1})H(\tau)d\tau=I$, the identity matrix.

The group $\mathfrak{G}$ acts by matrix multiplication on the space of all $n\times n$ matrices through the medium of the representation H. By Lemma 3.2 there is a closed neighborhood N_1 of the identity matrix and a closed subset C_1 of N_1 such that if $M\in N_1$ there is exactly one point of the form $H(\sigma)M$ which is in C_1. We may assume that N_1 contains only nonsingular matrices. Let $N=K^{-1}(N_1)$ (the complete inverse image of N_1 under K) and $C=K^{-1}(C_1)$. Then N is a closed neighborhood of p and C is a closed subset of N.

Suppose now that $q\in N$. Then $K(q)\in N_1$ and there is exactly one point of C_1 of the form $H(\sigma)K(q)$. Since $K(q)$ is nonsingular it can have only one representation in this form. Thus there is exactly one element $\sigma\in\mathfrak{G}$ such that $K(\sigma(q))=H(\sigma)K(q)\in C_1$; that is, there is exactly one element $\sigma\in\mathfrak{G}$ such that $\sigma(q)\in C$. This completes the proof.

3.4 THEOREM. *Let $\mathfrak{G}$ be a compact Lie group which acts on a completely regular space $\mathfrak{R}$. Let p be a point of $\mathfrak{R}$ and suppose that $\mathfrak{G}_q$ is conjugate to $\mathfrak{G}_p$ for all $q\in\mathfrak{R}$. Then there exists a closed neighborhood N of p and a closed subset C of N such that the orbit of every point of N has exactly one point in common with C.*

PROOF. Let S be the subset of $\mathfrak{R}$ consisting of all points left fixed by every element of $\mathfrak{G}_p$. S is closed. If $s\in S$, then $\mathfrak{G}_s\supset\mathfrak{G}_p$, but in a compact Lie group no subgroup contains one of its conjugates properly, hence $\mathfrak{G}_s=\mathfrak{G}_p$. If $q\in\mathfrak{R}$, $\mathfrak{G}_q=\sigma\mathfrak{G}_p\sigma^{-1}$ for some $\sigma\in\mathfrak{G}$. Then $\mathfrak{G}_{\sigma^{-1}(q)}=\sigma^{-1}\mathfrak{G}_q\sigma=\mathfrak{G}_p$; hence $\sigma^{-1}(q)\in S$ and $q\in\mathfrak{G}(S)$. We have proved that $\mathfrak{G}(S)=\mathfrak{R}$; moreover, for any set $F\subset\mathfrak{R}$, $\mathfrak{G}(F)=\mathfrak{G}(S\cap\mathfrak{G}(F))$.

Let $\mathfrak{H}$ be the normalizer of $\mathfrak{G}_p$ in $\mathfrak{G}$. $\mathfrak{H}$ is a closed[4] subgroup of $\mathfrak{G}$. $\mathfrak{H}$ maps S into itself, for, if $\sigma\in\mathfrak{H}$ and $s\in S$, then $\mathfrak{G}_{\sigma(s)}=\sigma\mathfrak{G}_s\sigma^{-1}=\sigma\mathfrak{G}_p\sigma^{-1}=\mathfrak{G}_p$, giving $\sigma(s)\in S$. Conversely, suppose that s, $t\in S$ and

[4] It can be verified directly that, in any topological group, the normalizer of a closed subgroup is closed.

that $\sigma(s)=t$. Then $\mathfrak{G}_p=\mathfrak{G}_t=\sigma\mathfrak{G}_s\sigma^{-1}=\sigma\mathfrak{G}_p\sigma^{-1}$, showing that $\sigma\in\mathfrak{H}$. From this we derive the formula: if $E\subset S$, $\mathfrak{H}(E)=S\cap\mathfrak{G}(E)$.

Since every element of S is left fixed by $\mathfrak{G}_p$ and S is mapped into itself by $\mathfrak{H}$, $\mathfrak{H}/\mathfrak{G}_p$ may be thought of as a compact Lie group acting on the space S. As a subspace of a completely regular space, S is completely regular. No element of S is left fixed by any element of $\mathfrak{H}/\mathfrak{G}_p$ except the identity. By Theorem 3.3, there is a closed neighborhood N_1 of p in S and a closed subset C of N_1 such that if $q\in N_1$, then $(\mathfrak{H}/\mathfrak{G}_p)(q)=\mathfrak{H}(q)$ intersects C in exactly one point. Let $N=\mathfrak{G}(N_1)$. Suppose that $r\in N$. Then $r=\sigma(q)$ for some $\sigma\in\mathfrak{G}$ and $q\in N_1$, hence $\mathfrak{G}(r)=\mathfrak{G}(q)$. Since $C\subset S$, $C\cap\mathfrak{G}(r)=C\cap S\cap\mathfrak{G}(q)=C\cap\mathfrak{H}(q)$, which is a one element set. As a closed subset of the closed set S, C is closed in $\mathfrak{R}$. We have now proved everything required except that N is a closed neighborhood of p. We already know that it is closed by Lemma 2.1.

To show that N is a neighborhood of p we must show that $p\notin\mathrm{Cl}(\mathfrak{R}-N)$ or since $p\in S$, that $p\notin S\cap\mathrm{Cl}(\mathfrak{R}-N)$. Using the formulas obtained in the first and second paragraphs of this proof, we have $\mathfrak{R}-N=\mathfrak{R}-\mathfrak{G}(N_1)=\mathfrak{G}(\mathfrak{R}-\mathfrak{G}(N_1))=\mathfrak{G}(S-S\cap\mathfrak{G}(N_1))=\mathfrak{G}(S-\mathfrak{H}(N_1))$. Using also the result of Lemma 2.2, $S\cap\mathrm{Cl}(\mathfrak{R}-N)=S\cap\mathrm{Cl}(\mathfrak{G}(S-\mathfrak{H}(N_1)))=S\cap\mathfrak{G}(\mathrm{Cl}(S-\mathfrak{H}(N_1)))=\mathfrak{H}(\mathrm{Cl}(S-\mathfrak{H}(N_1)))=\mathrm{Cl}(\mathfrak{H}(S-\mathfrak{H}(N_1)))=\mathrm{Cl}(S-\mathfrak{H}(N_1))\subset\mathrm{Cl}(S-N_1)$. Since N_1 is a neighborhood of p in S, $p\notin\mathrm{Cl}(S-N_1)$, a fortiori $p\notin S\cap\mathrm{Cl}(\mathfrak{R}-N)$, and we are through.

3.5 COROLLARY. *Let $\mathfrak{G}$ be a compact Lie group which acts on a completely regular space $\mathfrak{R}$. Let p be a point of $\mathfrak{R}$ such that $\mathfrak{G}_p$ is conjugate to $\mathfrak{G}_q$ for all q throughout a neighborhood of p. Then there exists a closed neighborhood N of p and a closed subset C of N such that the orbit of every point of N has exactly one point in common with C.*

PROOF. Let M be a closed neighborhood of p such that $\mathfrak{G}_q$ is conjugate to $\mathfrak{G}_p$ if $q\in M$. The set $R'=\mathfrak{G}(M)$ is a closed subset of $\mathfrak{R}$. If $r\in R'$, then $r=\sigma(q)$ where $\sigma\in\mathfrak{G}$ and $q\in M$. Then $\mathfrak{G}_r=\sigma\mathfrak{G}_q\sigma^{-1}$ which is a conjugate of $\mathfrak{G}_p$. We apply Theorem 3.4 to the space R' and observe that the set N which is a closed neighborhood of p in R' is also a closed neighborhood of p in $\mathfrak{R}$. Likewise, the set C, being closed in R', is closed in $\mathfrak{R}$.

3.6 THEOREM. *Let $\mathfrak{G}$ be a compact Lie group which acts on a completely regular space $\mathfrak{R}$. Let p be a point of $\mathfrak{R}$ such that $\mathfrak{G}_p$ is conjugate to $\mathfrak{G}_q$ for all q throughout a neighborhood of p. Then there exists a closed neighborhood of $\mathfrak{G}(p)$ which is fibered as a direct product by the orbits.*

PROOF. We have only to observe that the set C constructed in the proof of Theorem 3.4 satisfies the hypothesis of Theorem 2.3, while $\mathfrak{G}(C)=\mathfrak{G}(N)$ is a closed neighborhood of $\mathfrak{G}(p)$.

4. Application to topological groups. We shall now use the previous results to investigate the nature of coset decompositions in a topological group. Theorem 4.1 is an immediate corollary of Theorem 3.6, while Theorem 4.2 extends this result to a broader class of subgroups.

4.1 THEOREM. *Let $\mathfrak{R}$ be any topological group and let $\mathfrak{G}$ be a subgroup which is a compact Lie group. Then there is a neighborhood of $\mathfrak{G}$ which is fibered as a direct product by the cosets of $\mathfrak{G}$.*

PROOF. We may consider that the group $\mathfrak{G}$ acts on the space $\mathfrak{R}$ by setting $\sigma(\tau)=\sigma\tau$ where $\sigma\in\mathfrak{G}$ and $\tau\in\mathfrak{R}$. As a topological group, $\mathfrak{R}$ is a completely regular space. Since the orbits of points of $\mathfrak{R}$ are simply right cosets of $\mathfrak{G}$, the theorem follows directly from Theorem 3.6. The same result for left coset decompositions follows by considering the anti-isomorphism $\sigma\rightarrow\sigma^{-1}$ of $\mathfrak{R}$ with itself.

4.2 THEOREM. *Let $\mathfrak{R}$ be any topological group and let $\mathfrak{G}$ be a closed subgroup of $\mathfrak{R}$ which is a covering group of a compact Lie group. Then there is a closed neighborhood of $\mathfrak{G}$ which is fibered as a direct product by the cosets of $\mathfrak{G}$.*

PROOF. We shall construct a closed set C in $\mathfrak{R}$ such that the mapping χ: $(\sigma,\tau)\rightarrow\sigma\tau$ of $\mathfrak{G}\times C$ onto $\mathfrak{G}(C)$ is a homeomorphism, and $\mathfrak{G}(C)$ is a neighborhood of $\mathfrak{G}$.

Let $\mathfrak{K}$ be a discrete normal subgroup of $\mathfrak{G}$ such that $\mathfrak{H}=\mathfrak{G}/\mathfrak{K}$ is a compact Lie group. Let ψ be the natural map of $\mathfrak{G}$ onto $\mathfrak{H}$. $\mathfrak{K}$ is a discrete subgroup of $\mathfrak{R}$, hence we can form the homogeneous space $\mathfrak{R}/\mathfrak{K}$ of *right* cosets of $\mathfrak{K}$. Let ϕ be the natural map of $\mathfrak{R}$ onto $\mathfrak{R}/\mathfrak{K}$. Since $\mathfrak{K}$ is discrete we can choose a closed neighborhood S of ϵ, the identity of $\mathfrak{R}$, such that $SS^{-1}\cap\mathfrak{K}=\{\epsilon\}$. It follows from this that $S\cap\rho S$ is void if $\rho\in\mathfrak{K}$ unless $\rho=\epsilon$. Let $\phi(S)=S_1$. Then S_1 is a neighborhood of $\phi(\epsilon)$ in $\mathfrak{R}/\mathfrak{K}$ and ϕ maps S homeomorphically onto S_1. In fact, ϕ maps any set of the form $S\sigma$ ($\sigma\in\mathfrak{R}$) homeomorphically onto a neighborhood of $\phi(\sigma)$. Thus, since complete regularity is a local property, $\mathfrak{R}/\mathfrak{K}$ is a completely regular space.

Since $\mathfrak{K}$ is a normal subgroup of $\mathfrak{G}$, if $\sigma\in\mathfrak{G}$ and $\tau\in\mathfrak{R}$ we have $(\sigma\mathfrak{K})(\mathfrak{K}\tau)=\sigma\mathfrak{K}\tau=\mathfrak{K}\sigma\tau$. It follows that the mapping $(\psi(\sigma),\ \phi(\tau))\rightarrow\phi(\sigma\tau)$ of $\mathfrak{H}\times(\mathfrak{R}/\mathfrak{K})$ onto $\mathfrak{R}/\mathfrak{K}$ is well-defined. It is also continuous because of the continuity of the group operation and the definition of the topology $\mathfrak{H}$ and $\mathfrak{R}/\mathfrak{K}$. Thus the compact Lie group $\mathfrak{H}$ acts on the completely regular space $\mathfrak{R}/\mathfrak{K}$ under the convention $\psi(\sigma)(\phi(\tau))$

$=\phi(\sigma\tau)$. Furthermore, $\phi(\tau)=\psi(\sigma)(\phi(\tau))=\phi(\sigma\tau)$ only if $\sigma\in\mathfrak{K}$; that is, if $\psi(\sigma)$ is the identity of $\mathfrak{H}$. By Theorem 3.3, we can choose a closed set $C_0\subset\mathfrak{N}/\mathfrak{K}$ which intersects each orbit of a closed neighborhood of $\mathfrak{H}(\phi(\epsilon))$ in just one point. We may assume $\phi(\epsilon)\in C_0$ (if not, we could replace C_0 by $\sigma(C_0)$ where σ is an appropriate element of $\mathfrak{H}$). Let $C_1=S_1\cap C_0$. Then C_1 is a closed neighborhood of $\phi(\epsilon)$ in C_0. From the direct product representation of $\mathfrak{H}(C_0)$ (cf. Theorem 2.3) it follows that $\mathfrak{H}(C_1)$ is a closed neighborhood of $\mathfrak{H}(\phi(\epsilon))$.

Let $C=S\cap\phi^{-1}(C_1)$. Then C is closed in $\mathfrak{N}$ and ϕ maps C homeomorphically onto C_1. C intersects each orbit of $\mathfrak{G}(C)$ exactly once. For suppose that $\rho\sigma=\tau$ where σ, $\tau\in C$ and $\rho\in\mathfrak{G}$. Then $\phi(\tau)=\phi(\rho\sigma)=\psi(\rho)\phi(\sigma)$ where $\phi(\tau)$, $\phi(\sigma)\in C_1$. It follows that $\psi(\rho)$ is the identity of $\mathfrak{H}$ or that $\rho\in\mathfrak{K}$. Then $\tau\in S\cap\rho S$, where $\rho\in\mathfrak{K}$, but this set is void unless $\rho=\epsilon$. Hence $\sigma=\tau$. This implies that the mapping χ is one-to-one. Being a restriction of the group mapping, χ is certainly continuous. To show that χ is a homeomorphism we must show that it carries open sets into open sets. It is enough to show that all sufficiently small open sets containing a given point are carried into open sets.

Consider the composite mapping $(\sigma,\tau)\to(\psi(\sigma),\phi(\tau))\to\psi(\sigma)(\phi(\tau))=\phi(\sigma\tau)$ which carries $\mathfrak{G}\times C$ onto $\mathfrak{H}\times C_1$ onto $\mathfrak{H}(C_1)$. The first of these steps is a local homeomorphism and the second is a homeomorphism by Theorem 2.3. Since both steps carry open sets into open sets, the same is true of the composite which is simply $\phi\circ\chi$. Let $\sigma_0\in\mathfrak{G}$ and $\tau_0\in C$ be fixed and let V be an open neighborhood of $\sigma_0\tau_0$ so small that ϕ restricted to V is one-to-one. Let U be an open neighborhood of (σ_0,τ_0) so small that $\chi(U)\subset V$. Then $\chi(U)$ is open, for we know that $\phi\circ\chi(U)$ is open, and $\chi(U)=V\cap\phi^{-1}(\phi\circ\chi(U))$.

We have only to show that $\mathfrak{G}(C)$ is a closed neighborhood of $\mathfrak{G}$ and the theorem will be proved. This follows immediately from the equation $\mathfrak{G}(C)=\phi^{-1}(\mathfrak{H}(C_1))$.

SOCIETY OF FELLOWS, HARVARD UNIVERSITY

LOCAL CONNECTION IN LOCALLY COMPACT SPACES

M. H. A. NEWMAN

It was proved by Hurewicz[1] that a compact space which is both LC^1 and lc^n is LC^n. In the present paper the corresponding result for locally compact spaces is proved, (a) for uniform local connection, and (b) for relative local connection.[2] The extension of Hurewicz's theorem to locally compact spaces is included in (b). The main difficulty in extending Hurewicz's methods is that his "Satz 6," on the passage from ϵ-homotopy to true homotopy, cannot be carried over to locally compact spaces without substantial modification, even when uniform local connection is assumed. To overcome this a stronger form of the lc^p and LC^p conditions is used, namely (for lc^p), the existence of a function $\zeta(\delta, x)$ such that, given a compact set F in the neighbourhood $U(x, \zeta(\delta, x))$ of any point x, there is a compact subset F' of $U(x, \delta)$ such that every q-cycle in F bounds in F', for $0 \leqq q \leqq p$; and analogously for LC^p. It is shown that these are equivalent to the ordinary lc^p and LC^p properties in locally compact (metric) spaces.

1. **Definitions.** It is assumed once for all that the containing space X is locally compact, and has metric ρ.[3] Homologies are relative to integral coefficients; cycles in X are Vietoris-cycles (but finite cycles are chains on some simplicial complex with vertices in X). The statement that Γ bounds in E means that Γ bounds in a compact subset of E. p denotes an integer not less than 0.

The letter F, with various suffixes, always denotes a compact set. If G is open, the statement "$F \subseteq G$ with a margin α" means that $\alpha > 0$, and $\mathrm{Cl}(U(F, \alpha))$ is compact and contained in G.[4] The existence of margins for every such F and G is ensured by the local compactness of X.

A set E_1 is *ac^p rel.* E_2 ("acyclic up to p rel. E_2") if every q-cycle in E_1 bounds in E_2, for $0 \leqq q \leqq p$. E_1 is *as^p rel.* E_2 ("aspherical up to p rel. E_2") if every mapping of the q-sphere S^q into E_1 is null-homotopic in E_2, for $0 \leqq q \leqq p$. The set E_1 is *strongly* ac^p (or *strongly* as^p) rel. E_2 if, given any F in E_1 there is an F' in E_2 such that F is ac^p (or as^p) rel. F'.

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[1] Hurewicz [4], denoted hereafter by H. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Problem 4 of Eilenberg and Wilder [3] is thereby settled affirmatively.

[3] Use is made at one point (Theorem 1, (B)) of a local separability condition.

[4] Cl. (X) denotes the closure of X.

Among the various definition of lc^p and ulc^p that are in the field we choose those which impose the heaviest conditions on the bounding cycles, and therefore the lightest conditions on the space. If $E_1 \subseteq E_2 \subseteq X$, E_1 is lc^p rel. E_2 if there is a positive function $\eta(x, \delta)$ such that $E_1 U(x, \eta(x, \epsilon))$ is ac^p rel. $E_1 U(x, \epsilon)$ for all points x of E_2 and any positive ϵ;[5] and X is (absolutely) lc^p if it is lc^p rel. X. The space X is ulc^p if there is a positive function $\eta(\epsilon)$ such that for all points x of X, $U(x, \eta(\epsilon))$ is ac^p rel. $U(x, \epsilon)$.

2. **Homology.** We consider *chain-realisations* of (abstract simplicial) complexes, in the sense of Lefschetz [7] and Begle [1]. The complexes realised all have their vertices in X, and every realisation t of a complex K is to satisfy $t(\sigma^0) = \sigma^0$, for all vertices σ^0 of K. If C is a finite chain on a complex in X, a *realisation of* C means a chain $t(C)$, where t is a realisation of the carrier complex $\|C\|$.[6]

If X is connected, a realisation of any complex can, by an arbitrarily small displacement of the vertices, be so modified that accidental clashes are avoided, that is, any common vertex of $\|t(\sigma)\|$ and $\|t(\sigma')\|$ belongs to some $\|t(\sigma'')\|$, where σ'' is a common sub-cell of σ and σ'. It will be assumed that this is always arranged. There is, then, for any vertex x of $\|t(C^p)\|$, a unique simplex σ of lowest dimension in $\|C^p\|$, such that $x \in \|t(\sigma)\|$. This σ may be called the C^p-carrier of x.

THEOREM 1. *If X is lc^p and $\alpha > 0$, and if F is a compact subset of X, there exists a finite set of p-cycles $\Gamma_1^p, \Gamma_2^p, \cdots, \Gamma_k^p$ in $U(F, \alpha)$ such that every Γ^p in $F \sim \sum_{i=1}^{k} n_i \Gamma_i^p$ in $U(F, \alpha)$, for suitable integers n_i.*[7]

Since X is 0-lc its components are open sets, and therefore the compact set F meets only a finite number of them. It is clearly sufficient to prove the theorem for each separate component meeting F, that is, we may assume X to be connected.

The theorem is proved by combining the following results.

(A). *Given a compact set F_1 there exists a positive function* $\lambda(\epsilon)$

[5] Cf. Eilenberg and Wilder [3] for the corresponding homotopy property.

[6] Note that the definition of a *partial* realisation t of K requires $t(\sigma^0)$ to be defined for all 0-cells σ^0 of K. The *norm* of a full or partial realisation t of K is max $\rho(x, y)$ for $x \in \|t(\sigma_i)\|$, $y \in \|t(\sigma_j)\|$, where σ_i and σ_j are subcells of the same cell σ of K. The *mesh* of t is max $\Delta\|\sigma'\|$ for all simplexes σ' of chains $t(\sigma)$. (ΔE = diameter of E.)

[7] Wilder [8] (see also Begle [2, Corollary 2.3]) has proved that when homologies are mod m, the conditions of Theorem 1 imply that at most a finite number of cycles of F are independent in $U(F, \alpha)$. The analogous result with integral coefficients is not strong enough for present purposes, since it would allow, for example, an infinite base (Γ_i^p) with $\Gamma_i^p \sim 2\Gamma_{i+1}^p$.

such that any partial chain-realisation t_0 of a complex of dimension not greater than $p+1$ of norm less than $\lambda(\epsilon)$ in F_1 can be extended to a full realisation in X of norm less than ϵ; and there is a positive function $\kappa(\delta, \epsilon)$ such that if mesh $t_0 < \kappa(\delta, \epsilon)$, mesh t can be made less than δ.

This is Begle [2], Theorem 2.1.[8] (Note that when t_0 is defined only for vertices the κ-condition is automatically satisfied.)

The α of the enunciation of Theorem 1 may be supposed such that $\mathrm{Cl}(U(F, \alpha))$ is compact. Let $\sum_0^\infty \epsilon_n$ be a positive series with sum less than α, such that $\epsilon_{n+1} < \epsilon_n/3$ and, taking $F_1 = \mathrm{Cl}(U(F, \alpha))$ in (A), $\epsilon_{n+1} < \lambda(\epsilon_n)$.

(B) *Every finite cycle C_0^p in F of mesh less than ϵ_1 is the first member of a projection-cycle $\{C_n^p\}$ in $U(F, \alpha)$, the projection ϕ_n: $C_{n+1}^p \to C_n^p$ being an ϵ_n-projection.*[9]

Let $\eta_n = \sum_0^n \epsilon_r$, and make the inductive hypothesis that a finite cycle C_n^p of mesh less than ϵ_{n+1} is defined in $U(F, \eta_{n-1})$ (for $n = 0$, in F). Since $U(F, \eta_{n-1}) \subseteq U(F, \alpha) \subseteq F_1$, there is by (A) a chain-realisation C_{n+1}^p of C_n^p, of mesh less than ϵ_{n+2} and norm less than ϵ_n, and hence contained in $U(F, \eta_{n-1} + \epsilon_n) = U(F, \eta_n)$. This justifies the recursive definition of C_n^p. For each vertex x of C_{n+1}^p take $\phi_n(x)$ to be any vertex of the C_n^p-carrier of x (defined above). Then ϕ_n is an ϵ_n-projection.

Let the sequence (ϵ_n) satisfy the conditions of (B), and also $\epsilon_{n+3} < \kappa(\epsilon_{n+2}, \epsilon_n)$ (F_1 being as before).

(C) *Let $\Gamma_r^p = (Z_{rn}^p)$ be, for $r = 1, 2$, a p-cycle in X. Sufficient conditions for $\Gamma_1^p \sim \Gamma_2^p$ in $U(F, \alpha)$ are*

$$Z_{rn}^p \sim_{\epsilon_{n+3}} Z_{r,n+1}^p \text{ in } U(F, \eta_n) \qquad \text{for } n \geqq 0 \tag{1}$$

and

$$Z_{10}^p \sim_{\epsilon_2} Z_{20}^p \text{ in } U(F, \epsilon_0). \tag{2}$$

Let $Z_{r,n+1}^p - Z_{rn}^p = \beta Y_{rn}^{p+1}$ (= boundary of Y_{rn}^{p+1}), where Y_{rn}^{p+1} is a chain of mesh less than ϵ_{n+3} in $U(F, \eta_n)$ (whence mesh $Z_{rn}^p < \epsilon_{n+3}$); and let $Z_{10}^p - Z_{20}^p = \beta D_0^{p+1}$, where D_0^{p+1} is of mesh less than ϵ_2, and $D_0^{p+1} \subseteq U(F, \epsilon_0)$. Assume inductively that for some $n \geqq 0$, a finite chain D_n^{p+1} has been defined so that $\beta D_n^{p+1} = Z_{1n}^p - Z_{2n}^p$, mesh $D_n^{p+1} < \epsilon_{n+2}$, and $D_n^{p+1} \subseteq U(F, \eta_n)$. Then $Q_n^{p+1} = D_n^{p+1} + Y_{1n}^{p+1} - Y_{2n}^{p+1}$ is a $(p+1)$-chain with boundary $Z_{1,n+1}^p - Z_{2,n+1}^p$, and $\|Q_n^{p+1}\| \subseteq U(F, \eta_n) \subseteq F_1$. A partial realisation, t_0, of $\|Q_n^{p+1}\|$ is determined by putting $t_0(\sigma) = \sigma$ if[10] $\sigma \in \|Q_n^0\|$

[8] Our definitions are slightly different, but the proof is almost exactly similar.

[9] Cf. Begle [1, Lemma 2.4]. The property asserted of ϕ_n means that the projection-prism has mesh less than ϵ_n, and hence mesh $C_n^p < \epsilon_{n+1}$.

[10] $\|Q_n^0\|$ = set of vertices of $\|Q_n^{p+1}\|$, and in general K^m = set of cells of K of dimensions not greater than m.

$\bigcup \|Z^p_{1,n+1}\| \cup \|Z^p_{2,n+1}\|$. Mesh $t_0 < \epsilon_{n+4} < \kappa(\epsilon_{n+3}, \epsilon_{n+1})$, and norm $t_0 \leqq$ mesh $\|\phi^{p+1}_n\| < \epsilon_{n+2} < \lambda(\epsilon_{n+1})$. Hence t_0 can be extended to give a realisation D^{p+1}_{n+1} of Q^{p+1}_n, of mesh less than ϵ_{n+3}, and norm less than ϵ_{n+1}. Hence

$$D^{p+1}_{n+1} \subseteq U(F, \eta_n + \epsilon_{n+1}) = U(F, \eta_{n+1});$$

and

$$\beta D^{p+1}_{n+1} = t(\beta Q^{p+1}_n) = t(Z^p_{1,n+1} - Z^p_{2,n+1}) = Z^p_{1,n+1} - Z^p_{2,n+1}.$$

The recursive definition of D^{p+1}_n is justified, and (C) is therefore proved.

PROOF OF THEOREM 1. Let $\{U(x_i, \epsilon_4/6\}$ be a finite covering of F, with $x_i \in F$, and let N be the nerve of the covering $\{U(x_i, \epsilon_4/2)\}$, with the points x_i as vertices. Choose a p-dimensional basis of homology $C^p_{(0}$ $(i=1, 2, \cdots, k)$ of N. Since mesh $N < \epsilon_4$, (B) is applicable, with the series $(\epsilon_0, \epsilon_1, \cdots)$ replaced by $(\epsilon_3, \epsilon_4, \cdots)$, to give a projection-cycle Γ^p_i in $U(F, \alpha)$ with first member $C^p_{(0}$. *The cycles* Γ^p_i *are the required set.* For let Γ^p be any p-cycle in F, and (Z^p_n) $(n=0, 1, \cdots)$ a subsequence of its members satisfying

(a) $Z^p_{n+1} \sim_{\epsilon_{n+2}} Z^p_n$ in F, (b) mesh $Z^p_0 < \epsilon_4/6$.

If, for each x of F, $\theta(x)$ is a vertex of N in $U(x, \epsilon_4/6)$, and if x and y are vertices of the same cell of Z^p_0, $\rho(\theta x, \theta y) < \epsilon_4/6 + \rho(x, y) + \epsilon_4/6 < \epsilon_4/2$, and hence θ is a simplicial mapping of Z^p_0 into N.

Let $\theta(Z^p_0) = Z'^p \sim \sum_1^k n_i C^p_{(0}$ in N, for some n_i. *The pair of cycles* Γ^p *and* $\sum_1^k n_i \Gamma^p_i$ *satisfy the conditions* (1) *and* (2) *of* (C). Let $\Gamma^p_i = \{C_{0i}, C_{1i}, C_{2i}, \cdots\}$. *Condition* (1). For Γ^p this follows from (a) above. The chain $\sum n_i \Gamma^p_i$ is a projection-cycle for which ϕ_n is an ϵ_{n+3}-projection, and all vertices of $\| \sum n_i C_{ni} \|$ belong to $\| \sum n_i C_{n+1,i} \|$.[11] Hence condition (1) is satisfied if $\|C_{n+1,i}\| \subseteq U(F, \eta_n)$. This is so, since $\|C_{n+1,i}\| \subseteq U(F, \sum_3^{n+3} \epsilon_r)$ (proof of (B), ϵ_{r+3} replacing ϵ_r), and $\sum_3^{n+3} \epsilon_r < \sum_0^n \epsilon_r = \eta_n$. *Condition* (2). The θ-prism joining to Z^p_0 Z'^p has mesh less than $\epsilon_4/2 < \epsilon_2$, whence $Z^p_2 \sim_{\epsilon_2} Z'^p$ in F; and $Z'^p \sim \sum n_i C^p_{(0}$ in N, a complex of mesh less than ϵ_2 in F.

THEOREM 2. *If* X *is* lc^p, *an open set* G_1 *which is* ac^p *rel. an open set* G_2 *is also strongly* ac^p *rel.* G_2.

Suppose $F \subseteq G_1$ with a margin α. For $0 \leqq q \leqq p$, let $\Gamma^q_1, \Gamma^q_2, \cdots, \Gamma^q_{k_q}$ be a basis of q-cycles in $U(F, \alpha)$ constructed as in Theorem 1. Let Γ^q_i

[11] By the general rule that $t(\sigma^0) = \sigma^0$, above.

bound in the subset F_{qi} of G_2. Then every q-cycle in F bounds in the compact subset

$$F' = \mathrm{Cl}(U(F, \alpha)) \cup \bigcup_{q=0}^{p} \bigcup_{i=1}^{k_q} F_{qi}$$

of G_2.

3. **Homotopy.** The relation of homotopy is denoted by $\simeq$ and ϵ-homotopy[12] by $\simeq_\epsilon$. S^p is the sphere $\sum_0^p \xi_r^2 = 1$ in R^{p+1}; S^r is, for $r<p$, the intersection of S^{r+1} with $\xi_{r+1}=0$; c_0 is the point $(1, 0, \cdots, 0)$ of R^{p+1}. A set E_1 is ϵ-as^p *rel.* E_2 if, for $0 \leqq q \leqq p$, every mapping $f: S^q \to E_1 \simeq_\epsilon 0$ in E_2 rel. c_0.

THEOREM 3. *Let the open set G be LC^{p-1} rel. X[13] and suppose that a positive function $\eta_0(\delta, x)$ exists with this property: to any point x, and any compact F in $GU(x, \eta_0(\delta, x))$ there corresponds a compact F' in $GU(x, \delta)$ such that F is ϵ-as^p rel. F' for every positive ϵ. Then $GU(x, \eta_0(\delta, x))$ is strongly as^p rel. $GU(x, \delta)$; and therefore G is LC^p rel. X. (The LC^{p-1} condition is vacuously satisfied if $p=0$.)*

COROLLARY. *If $G=X$ and $\eta_0(\delta, x)=\eta_0(\delta)$, independent of x, X is p-ULC.*

This theorem replaces "Satz 6" of Hurewicz [4] in locally compact spaces. Although our proof follows his closely, the many changes of detail make it necessary to give the full proof, which depends on the following lemmas D, E, and F.

(D) ($p \geqq 1$) *Given a compact set F in a locally compact LC^{p-1} space X, and a positive number ϵ, there exists a positive $\eta_2=\eta_2(\epsilon, F)$ with the following property: if P is a polyhedron, and Q a subpolyhedron[14] of P^{p-1}, and if f_0, f_1 map P into F and satisfy $\rho(f_0, f_1)<\eta_2$, there exists a mapping $f_0': P \to X$, agreeing with f_0 on Q, and deformable into f_1 within ϵ, in X.*[15]

The proof of (D) is omitted, since only obvious changes are needed in the proofs of H, Sätze 1–3.

Let G be as in Theorem 3. Let $F_0 \subseteq G$ with a margin α_0, and let $0<\delta \leqq \alpha_0$. There exists a finite covering $\{U_i\} = \{U(x_i, \eta_0(\delta, x_i))/2\}$

[12] See H, p. 477: $f_1 \simeq_\epsilon f_2$ if f_1 and f_2 are connected by an ϵ-chain of points in the space of mappings.

[13] This is the homotopy-local-connection introduced by Lefschetz [5]. For relative local connection see Eilenberg and Wilder [3].

[14] No distinction is made in terminology between a polyhedral complex P and the polyhedron which is its locus; but the corresponding abstract simplicial complex determined by the vertices of P is to be distinguished from P. It is denoted by $\|P\|$.

[15] That is, f_0' and f_1 are connected in the space X^P by an arc of diameter less than ϵ.

of the compact set F_0, with $x_i \in F_0$. Since $\eta_0(\delta, x_i) \leqq \delta \leqq \alpha_0$, $\bigcup U_i \subseteq G$. Let $\eta_3 = \min_i \eta_0(\delta, x_i)$.

(E) *($p \geqq 1$) If f_1, f_2 map the p-element E^p (with boundary S^{p-1}) into F_0, and if $f_1 \big| S^{p-1} = f_2 \big| S^{p-1}$ and $\Delta f_r(E^p) < \eta_3/2$, for $r = 1, 2$, then $f_1 \simeq_\epsilon f_2$ with fixed S^{p-1}, in $U(x_i, \delta)$, for some i.*

If $y \in f_1(S^{p-1})$, one of the points x_i satisfies $\rho(x_i, y) < \eta_0(\delta, x_i)/2$ and therefore $f_r(E^p) \subseteq U(x_i, \eta_3/2 + \eta_0(\delta, x_i)/2) \subseteq U(x_i, \eta_0(\delta, x_i))$ for $r = 1, 2$. Hence $f_1 \simeq_\epsilon f_2$ in $U(x_i, \delta)$, by the conditions of Theorem 3 and H, Satz 4.

We suppose that for each compact F_0 and each positive δ, such a covering $\{U_i\}$ is chosen, and denote the corresponding $\min_i \eta_0(\delta, x_i)$ by $\eta_3(\delta, F_0)$. Let $\mu(x, y)$ be defined as in H.[16]

(F) *Given any compact F_0 in G there exists $\eta_4(\delta, F_0) > 0$ such that if f_1, f_2 map S^p into F_0 and $\rho(f_1, f_2) < \eta_4(\delta, F_0)$, then $\mu(f_1, f_2) < \delta$.* (The distances $\rho(f_1, f_2)$ and $\mu(f_1, f_2)$ are in the function space X^{S^p}.)

First suppose $p > 0$. Let α be a margin of F_0 rel. G. We define:

$$\delta' = \min(\delta, \alpha), \qquad F_1 = \mathrm{Cl}\,(U(F_0, \alpha)),$$

$$\xi = \frac{1}{6}\eta_3\left(\frac{1}{4}\delta', F_1\right), \qquad \eta_4(\delta, F_0) = \eta_2(\xi, F_0),$$

the functions η_2, η_3 having the meanings given above. Let two mappings f_1, f_2 of S^p into F_0 be given, satisfying $\rho(f_1, f_2) < \eta_4(\delta, F_0)$. Let S^p be simplicially subdivided into a polyhedron Σ^p, so finely that $\Delta f_r(\sigma^p) < \xi$ for each (continuous) p-simplex σ^p of Σ^p, $r = 1, 2$. Then by (D), a mapping $f_1' : S^p \to X$ exists, agreeing with f_1 on Σ^{p-1}, and such that $f_2 \simeq f_1'$ within ξ in G. Hence $\mu(f_1', f_2) < \xi < \delta/2$. Also, since $\delta' \leqq \alpha$ $f_1'(S^p) \subseteq U(F_0, \alpha/2) \subseteq F_1$; and $\rho(f_1', f_2) \leqq \mu(f_1', f_2) < \xi$. Therefore $\Delta f_1'(\sigma^p) < 3\xi = \eta_3(4^{-1}\delta', F_1)/2$ for every σ^p of Σ^p. By Lemma (E) it follows that $f_1' \big| \sigma^p \simeq_\epsilon f_1 \big| \sigma^p$, with fixed $\beta\sigma^p$, in a set of diameter $\delta/2$, for every positive ϵ. Since this holds for every σ^p of Σ^p, $\mu(f_1', f_1) < \delta/2$. Thus $\mu(f_1, f_2) \leqq \mu(f_1, f_1') + \mu(f_1', f_2) < \delta$.

If $p = 0$ let $\eta_4(\delta, F_0) = \eta_3(\delta', F_0)/2$. (The definition of η_3 remains significant when $p = 0$.) It is sufficient in this case to show that if $x, y \in F_0$ and $\rho(x, y) < \eta_4(\delta, F_0)$, then x and y are joined by an ϵ-chain of points of G, of diameter less than δ. There exists a point x_i such that $\rho(x, x_i) < \eta_0(\delta', x_i)/2$. Since $\rho(x, y) < \eta_3/2 \leqq \eta_0(\delta', x_i)/2$, x and y are both in $U(x_i, \eta_0(\delta', x_i)) \subseteq U(x_i, \alpha) \subseteq G$. Hence the required chain exists, by the conditions of Theorem 3.

Theorem 3 can now be proved. Let a positive δ, a point x of X, and a compact F in $GU(x, \eta_0(\delta, x))$ be given, and let F' be a set as in the

[16] In any metric space R, $\mu(x, y)$ is the greatest lower bound of diameters of subsets of R in which x and y are ϵ-connected for every positive ϵ.

enunciation of Theorem 3. Then F *is as*^{p} *rel.* $F_2 = \mathrm{Cl}(U(F', \beta))$, where β is any margin of F' rel. $GU(x, \delta)$. Let $\delta_n = \eta_4(\beta/2^n, F_2)$, let x_0 be a point of F, and let f map S^p into F. By the conditions of the theorem, the points f and $[x_0]$ (=constant function x_0) of F^{S^p} are joined by a δ_1-chain, L_1, of points, all lying in F'^{S^p}. Assume inductively that $L_1, \cdots, L_n$ have been determined, L_r being formed by joining each consecutive pair of points of L_{r-1} by a δ_r-chain; and that the mappings which are the "points" of L_n all map S^p into $U(F', \sum_1^{n-1}\beta/2^r) \subseteq F_2$ (into F' when $n=1$). Two consecutive points f_r, f_s of L_n satisfy $\rho(f_r, f_s) < \delta_n = \eta_4(\beta/2^n, F_2)$; and hence by Lemma (F), $\mu(f_r, f_s) < \beta/2^n$, that is, f_r and f_s can be joined by a δ_{n+1}-chain of total diameter less than $\beta/2^n$ in the function space. This justifies the recursive definition of L_n.

It now follows, exactly as in H, p. 481, that $\mathrm{Cl}(\bigcup_1^\infty L_n)$ is the locus of a continuous path joining f to $[x_0]$ in $F_2^{S^p}$. The proof of Theorem 3 is thus completed.

4. **Homology and homotopy.** Theorem 4, general case ($p \geqq 2$). *If* G *is* LC^1 *rel.* X *and* lc^p *rel.* X *there is a positive function* $\zeta(\delta, x)$ *such that* $GU(x, \zeta(\delta, x))$ *is strongly as*^{p} *rel.* $GU(x, \delta)$ *for all* x *of* X.

Case 0: *Put* $p=0$ *and omit "*LC^1 *rel.* X *and."*

Case 1: *Put* $p=1$ *and omit "and* lc^1 *rel.* X*."*

Immediate corollaries of this theorem are

THEOREM 4.1 ($p \geqq 2$). *If* G *is* LC^1 *and* lc^p *it is* LC^p, *all rel.* X.

(When $G=X$ this is the generalisation of Hurewicz's theorem to locally compact spaces.)

THEOREM 4.2 ($p \geqq 0$). *If* G *is* LC^p *rel.* X *there exists a positive function* $\zeta(\delta, x)$ *such that* $GU(x, \zeta(\delta, x))$ *is strongly as*^{p} *rel.* $GU(x, \delta)$ *for all* x *of* X.

PROOF OF THEOREM 4, CASE 0. Let $\eta(\delta, x)$ be such that $GU(x, \eta(\delta, x))$ is ac^0 rel. $GU(x, \delta)$, and let $F \subseteq GU(x, \eta(\delta, x))$. By Theorem 2 there is an F' in $GU(x, \delta)$ such that F is ac^0 rel. F'. This implies that for any positive ϵ any two points of F are joined by an ϵ-chain in F', that is, that F is ϵ-as^0 rel. F' for every positive ϵ. Thus the conditions of Theorem 3 (0) are satisfied, if η_0 is replaced by η.

PROOF OF THEOREM 4, CASE 1. This is contained in the following theorem.

THEOREM 5. *If* X *is* LC^1, *any open set* G_1 *which is as*^{1} *rel. an open set* G_2 *is also strongly as*^{1} *rel.* G_2.

Let $F \subseteq G_1$ with margin α_0. Then there exist positive α_1 and α_2 such that, for $r=0, 1$, if $x \in F$, $U(x, \alpha_{r+1})$ is as^1 rel. $U(x, \alpha_r/3)$. Let $\{U(x_i, \alpha_2/6)\}$ be a finite covering of F ($x_i \in F$), and let N be the nerve of the covering $\{U(x_i, \alpha_2/2)\}$ realised in X, with the x_i as vertices. If then P is a polyhedron abstractly isomorphic with N, the mapping g_0 of P^0 into N^0 determined by the isomorphism can be extended to a mapping g_1 of P^1 into $U(F, \alpha_1/3)$.

Let f be a re-entrant path in F, that is, $f: \langle 0, 1\rangle \to F$ with $f(0) = f(1) = x_0 = g_0(z_0)$, say; and let the points $0 < \tau_1 < \tau_2 < \cdots < \tau_k = 1$ divide f into sub-paths s_j of diameter less than $\alpha_2/6$. Now each $f(\tau_j)$ is in some $U(y_j, \alpha_2/6)$, where $y_j \in N^0$ ($y_0 = y_k = x_0$), and a path γ_j of diameter less than $\alpha_1/3$ therefore runs from y_j to $f(\tau_j)$. By the usual process of extruding "tails" the path $f = \sum s_j$ is deformable in $U(F, \alpha_1/3)$ into

$$\sum_{j=1}^{k} (\gamma_{j-1} + s_j - \gamma_j)$$

the $\sum$ and $+$-signs denoting the usual path-summation. Now $\rho(y_{j-1}, y_j) < \alpha_2/6 + \Delta s_j + \alpha_2/6 < \alpha_2/2$ and therefore y_{j-1}, y_j are the g_1-images of the ends of a 1-cell σ_j^1 of P^1. Since $g_1(\sigma_j^1)$ and $\gamma_{j-1} + s_j - \gamma_j$ are both in $U(y_j, \alpha_1)$, the path $\gamma_{j-1} + s_j - \gamma_j$ is deformable in $U(y_j, \alpha_0)$, with fixed end points, into the path $g_1 | \sigma_j^1$. Thus $f \simeq g_1(s)$ in $U(F, \alpha_0)$, where s is a path on P with $s(0) = s(1) = z_0$.

Let $P_1, P_2, \cdots, P_l$ be the components of P, and let the paths $a_{1r}, a_{2r}, \cdots, a_{m_r r}$ in P_r^1 be representatives of a base of the fundamental group of P_r^1. The path s lies in one component, say P_r^1, and

$$g_1 s \simeq \pm g_1 a_{n_1 r} \pm g_1 a_{n_2 r} \pm \cdots \pm g_1 a_{n_k r}$$

on $g_1(P_r^1) \subseteq U(F, \alpha_1) \subseteq G_1$. By hypothesis, for each i and r, $g_1 a_{ir} \simeq 0$ in G_2, and therefore in a compact set F_{ir} in G_2. Hence

$$f \simeq 0 \text{ in } \mathrm{Cl}(U(F, \alpha_0)) \cup \bigcup_{1}^{l} \bigcup_{1}^{m_r} F_{ir}$$

a compact subset of G_2 independent of f.

PROOF OF THEOREM 4, GENERAL CASE ($p \geqq 2$). We make the inductive assumption that $4(p-1)$ is proved, and may therefore, by 4.1 $(p-1)$ and 4.2 $(p-1)$, assume that G is LC^{p-1} rel. X. Let $\zeta'(\delta, x)$ be the function corresponding to ζ in the dimension $p-1$, and let $\eta(\delta, x)$ be such that $GU(x, \eta(\delta, x))$ is ac^p rel. $GU(x, \delta)$ for every x. *Then* $\zeta(\delta, x)$ *may be put equal to* $\eta(\zeta'(\delta, x), x)$. This choice will be justified by Theorem 3 if it is shown that the condition of that theorem is satisfied, with η_0 replaced by ζ.

Let $F \subseteq GU(x, \zeta(\delta, x))$. By Theorem 2, there is an $F_1 \subseteq GU(x, \zeta'(\delta, x))$ (with a margin α) such that F is ac^p rel. F_1; and by Theorem $4(p-1)$ there is an $F' \subseteq GU(x, \delta)$ such that $\mathrm{Cl}(U(F_1, \alpha))$ is as^{p-1} rel. F'. *This is the set F' required in Theorem* 3.

Since X is LC^{p-1} and F_1 is compact, there is[17] a positive function $\eta_1(\xi)$ such that, given a polyhedron P^p, and any subpolyhedron Q containing all its vertices, any mapping $f_0: Q \to F_1$ whose continuous norm[18] is less than $\eta_1(\xi)$ can be extended to a mapping $f_1: P^p \to X$ of continuous norm less than ξ. Let a mapping $f: S^p \to F$ and a positive $\epsilon < \alpha$ be given. Divide S^p simplicially into a polyhedron Σ^p, and let C^p be a fundamental p-cycle on $\|\Sigma^p\|$.[14] The simplicial division is to be so fine that (a) $\Delta f(\sigma) < \eta_1(\epsilon/2)$ for every (continuous) simplex σ of Σ^p; and (b) there exist an abstract complex K^{p+1} containing $\|\Sigma^p\|$ as a subcomplex, a chain C^{p+1} on K^{p+1} with boundary C^p, and a mapping $f_1: K^{p+1} \to F_1$ of mesh less than $\eta_1(\epsilon/2)$, with $f_1 | \Sigma^0 = f | \Sigma^0$. That this is possible follows from the definition of F_1. Let P^{p+1} be a polyhedron such that $\|P^{p+1}\| = K^{p+1}$ and Σ^p is a subpolyhedron of P^{p+1}. The combination of f in S^p and f_1 at the vertices of K^{p+1} determines a continuous mapping of the subpolyhedron $\Sigma^p \cup P^0$ of P^p into F_1, of continuous norm less than $\eta_1(\epsilon/2)$. It can therefore be extended to a mapping $g_1: P^p \to X$ of continuous norm less than $\epsilon/2$. Thus g_1 is a mapping into $U(F_1, \epsilon/2) \subseteq \mathrm{Cl}(U(F_1, \alpha))$. From the definition of F' it follows that if $y_0 = g_1(c_0)$[19] $g_1 | P^{p-1} \simeq [y_0]$ in F'. Hence[20] there exists a mapping $g_2: P^p \to F'$ such that $g_1 \simeq g_2$ in F', and $g_2 | P^{p-1} = [y_0]$.

From this point on, the proof that $f \simeq_\epsilon 0$ (rel. c_0) in F' proceeds exactly like the remainder of the proof in H (pp. 484 and 485) that $f \simeq_\epsilon 0$ (rel. x_0) in U. The proof of Theorem $4(p)$ is thereby completed.

From the definition of ζ it is clear that if ζ' and η are independent of x, so also is ζ. The case $G = X$ is then of most interest, and gives the following theorems.

THEOREM 6. *If X is ULC^1 and ulc^p it is ULC^p, if $p \geqq 2$; and if X is ulc^0 it is ULC^0.*

THEOREM 6.1. *($p \geqq 0$). If X is ULC^p there exists a positive function $\zeta(\delta)$ such that $U(x, \zeta(\delta))$ is strongly as^p rel. $U(x, \delta)$ for all x of X.*

[17] Lefschetz [6, p. 120] = H, Satz (1a). The modifications needed to allow for X being only locally compact are obvious in view of the compactness of F_1.

[18] *Continuous norm of f_0* = least upper bound of $\rho(f_0(x), f_0(y))$ for x, y in the same cell of P^p.

[19] c_0 is the point $(1, 0, \cdots, 0)$ (cf. §3).

[20] H, Satz 2.

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University of Manchester

NATURAL HOMOMORPHISMS IN BANACH SPACES

J. DIEUDONNÉ

1. **Introduction.**[1] In a recent paper, published in this Bulletin [7], M. E. Munroe considers the following question. Let E be a Banach space, E^* its dual, G a closed linear subspace of E, Γ the subspace of E^* orthogonal to G (that is, the space consisting of the linear functionals x' on E such that $x'(x)=0$ for every $x\in G$); there is then a natural homomorphism T of E^* onto the dual G^*, with kernel Γ; Munroe investigates the continuity and openness of T when both E^* and G^* are given one of the following topologies: norm, weak, weak*, bounded weak, and bounded weak* (the topologies on E^* and G^* being always of the same type).

The openness of T for the norm topologies is proved by Krein and Smulian [5], and is an easy consequence of the Hahn-Banach extension theorem (see [3, p. 124, Theorem 17a]). Munroe proves the openness of T for the weak* and weak topologies (Theorems 3.3 and 3.4), but he has failed to notice that both theorems are particular cases of a general result I proved some years ago [3, p. 116, Theorem 7] for linearly convex spaces, by the same argument Munroe uses in his proof. I shall prove in this note that T is also open for the *bounded weak** and *bounded weak* topologies on E^* and G^*; also I shall rectify an error in the first statement of Munroe's Theorem 3.5.

2. **Characterization of the bounded weak* topology on E^*.** For $x\in E$ and $x'\in E^*$, I shall write $x'(x)=\langle x, x'\rangle$; for any subset A of E (resp. A' of E^*), A^0 (resp. A'^0) will be the subset of E^* (resp. E) consisting of the x' such that $|\langle x, x'\rangle|\leqq 1$ for every $x\in A$ (resp. the x such that $|\langle x, x'\rangle|\leqq 1$ for every $x'\in A'$); clearly A^0 is convex and weakly* closed, A'^0 convex and closed. In the following, the word "compact" is used in the Bourbaki sense (="bicompact").

LEMMA 1. *For every subset A of E, the set A^{00} is the smallest convex and closed subset of E containing both A and $-A$.*

It is clear that $A\subset A^{00}$ and $-A\subset A^{00}$; if A_1 is the smallest convex closed set containing A and $-A$, $A_1\subset A^{00}$. On the other hand, let x be a point in E not belonging to A_1; there exists a closed hyperplane H

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

separating x and A_1, by the Hahn-Banach theorem; this means that there is an $x' \in E^*$ such that $\langle y, x' \rangle \leqq 1$ for every $y \in A_1$, and $\langle x, x' \rangle > 1$; as $-A_1 = A_1$, one has also $|\langle y, x' \rangle| \leqq 1$ for every $y \in A_1$, and as $A \subset A_1$, $x' \in A^0$, and therefore $x \notin A^{00}$, which proves the lemma.

LEMMA 2 (MAZUR). *If A is strongly compact in E, so is A^{00}.*

A^{00} is the closure of the convex hull B of $A \cup (-A)$ in E, and therefore is complete for the metric on E; it is therefore enough to prove that B is strongly precompact. For every $\epsilon > 0$, there exist a finite number of points x_i in $A \cup (-A)$ $(1 \leqq i \leqq n)$ such that any $x \in A \cup (-A)$ is contained in one of the balls $\|x - x_i\| \leqq \epsilon$; if C is the convex hull of the finite set of the x_i, every point of B is at a distance not greater than ϵ from a point of C. But C is finite-dimensional and bounded, therefore compact; there are therefore a finite number of points y_j of C $(1 \leqq j \leqq n)$ such that any point of C is at a distance not greater than ϵ from one of the y_j; therefore every point of B is at a distance not greater than 2ϵ from one of the y_j, which completes the proof.

The bounded weak* topology on E^* may be defined as the finest topology on E^* which, on every ball $\|x'\| \leqq r$, coincides with the weak* topology. The open neighborhoods of the origin for that topology are therefore the sets U containing the origin and such that, for every ball S, $U \cap S$ is open for the topology induced on S by the weak* topology. We now prove the following theorem:

THEOREM 1. *A fundamental system of neighborhoods of the origin for the bounded weak* topology consists of the sets K^0, where K is an arbitrary strongly compact subset of E (in other words, the bounded weak* topology is identical with the k-topology in the sense of Arens* [1, p. 789]).

We shall see that this theorem is practically only a new formulation of a result of Banach [2, pp. 119–120]. Let us prove first that a set K^0 is a neighborhood of O for the bounded weak* topology. Let S be any ball $\|x'\| \leqq r$; as K is compact, there exist a finite number of points x_i in K $(1 \leqq i \leqq n)$ such that any point of K is at a distance not greater than $1/2r$ from one of the points x_i. Let W be the neighborhood of O in the weak* topology, defined by the n inequalities $|\langle x_i, x' \rangle| \leqq 1/2$; for any $x' \in W \cap S$, and any $x \in K$, there is a point x_i such that $\|x - x_i\| \leqq 1/2r$, and therefore

$$|\langle x, x' \rangle| = |\langle x_i, x' \rangle + \langle x - x_i, x' \rangle| \leqq \frac{1}{2} + \frac{1}{2r}\|x'\| \leqq 1$$

which proves that $W \cap S$ is contained in K^0.

We now prove conversely that if U is any open neighborhood of the origin for the bounded weak* topology, it contains a set K^0. This will be a consequence of the following lemma:

LEMMA 3. *There exists in E a finite set A_1 of points of norm greater than 1 and, for every $n>0$, a finite set B_n of points of norm not greater than $1/n$, such that, if $A_{n+1}=A_n\cup B_n$ for $n>0$, the intersection of A_n^0 with the ball $\|x'\|\leqq n$ is contained in U.*

Suppose that lemma is proved: the union A of all sets A_n and of the point O is obviously strongly compact; let S be any ball $\|x'\|\leqq r$, and n an integer greater than r; Lemma 3 shows that the intersection $S\cap A_n^0$ is contained in U, and *a fortiori* $S\cap A^0\subset U$; as the radius of S is arbitrary, $A^0\subset U$.

Lemma 3 is substantially the result of Banach quoted above; for the sake of completeness, we reproduce its proof. The intersection of U with the ball $\|x'\|\leqq 1$ contains a weak* neighborhood of O, and therefore there exists a finite set A_1 of E such that the intersection of A_1^0 with the ball $\|x'\|\leqq 1$ is contained in U; it may obviously be supposed that the norms of the elements in A_1 are greater than 1. Now argue inductively in the following way: suppose A_n is defined as a finite set in E such that the intersection of A_n^0 with the ball S_n: $\|x'\|\leqq n$ is contained in U, and let us show that there exists a finite set B_n of points of norm $\leqq 1/n$ such that the intersection of $(A_n\cup B_n)^0$ with the ball S_{n+1}: $\|x'\|\leqq n+1$ is contained in U. Suppose that proposition were not true, and let K be the intersection of S_{n+1} with the complement of U in E^*; K is closed for the bounded weak* topology, and therefore for the weak* topology, since on S_{n+1} both topologies coincide; as K is bounded, it is weakly* compact [3, p. 128, Theorem 22]. For every finite subset B of the ball $\|x\|\leqq 1/n$ there would exist a nonvoid subset $H(B)$ of K, contained in $(A_n\cup B)^0$; the subsets $H(B)$ constitute obviously a filter base on K, having therefore a cluster point x_0' in K for the weak* topology, since K is weakly* compact. For every $\epsilon>0$ and every $x\in E$ such that $\|x\|\leqq 1/n$, there exists therefore an $x'\in K$ such that $|\langle x, x'\rangle|\leqq 1$ and $|\langle x, x'-x_0'\rangle|\leqq\epsilon$, whence $|\langle x, x_0'\rangle|\leqq 1+\epsilon$, and as ϵ is arbitrary, $|\langle x, x_0'\rangle|\leqq 1$; as that relation holds for every x such that $\|x\|\leqq 1/n$, we have $\|x_0'\|\leqq n$. The same argument shows that, for every $x\in A_n$, one has $|\langle x, x_0'\rangle|\leqq 1$, in other words $x_0'\in A_n^0\cap S_n$, and the induction hypothesis implies $x_0'\in U$, which is absurd, since x_0' is in K.

THEOREM 2. *The natural homomorphism T is open for the bounded weak* topology.*

We shall adapt to our purpose an argument of Köthe [4, p. 27].

Let K be a strongly compact subset of E; as K^{00} is strongly compact and $(K^{00})^0 = K^0$, we may suppose that K is convex and symmetric by Lemma 2. Let U be the neighborhood of O in E^* for the bounded weak* topology, consisting of the elements x' such that $|\langle x, x'\rangle| < 1$ for every $x \in K$ (Theorem 1); we want to prove that $T(U)$ contains a neighborhood of the origin in G^* for the bounded weak* topology. We shall reach that goal by proving that $T(U)$ contains all elements $u' \in G^*$ such that $|\langle x, u'\rangle| < 1$ for every $x \in K \cap G$ ($K \cap G$ being strongly compact in G). As u' may be considered as the restriction to G of a linear functional x_0' defined in E, we have to prove that if x_0' is such that $|\langle x, x_0'\rangle| < 1$ for every $x \in K \cap G$, there exists a linear functional y_0' such that $x_0' - y_0' \in \Gamma$ and that $|\langle y, y_0'\rangle| < 1$ for every $y \in K$.

Consider in E^* the closed subspace H orthogonal to all elements of K, and let ϕ be the natural homomorphism of E^* onto E^*/H; for every element $\phi(x')$ of E^*/H, $\|\phi(x')\| = \sup_{x \in K} |\langle x, x'\rangle|$ is a norm on E^*/H, as is readily verified. Moreover, if f is any linear functional on E^*/H, continuous for the topology defined by that norm, $f \circ \phi$ is a linear functional on E^*, such that $|f(\phi(x'))|$ is bounded on K^0; as K is strongly compact in E, it is also weakly compact, and therefore, by Arens's theorem [1, p. 790, Theorem 2], there exists an element $z \in E$ such that $f(\phi(x')) = \langle z, x'\rangle$ identically.

Let d be the distance of $\phi(x_0')$ to the linear subspace $\phi(\Gamma)$ in the normed space E^*/H and suppose first $\phi(\Gamma)$ is not everywhere dense in E^*/H. The Hahn-Banach extension theorem proves the existence of a continuous linear functional f_0 on E^*/H, having a norm equal to 1, equal to 0 in the subspace $\phi(\Gamma)$, and such that $f_0(\phi(x_0')) = d$. Let x_0 be an element of E such that $f_0(\phi(x')) = \langle x_0, x'\rangle$ identically; the definition of f_0 shows that $x_0 \in G$, and that, for every $x' \in E^*$, $|\langle x_0, x'\rangle| \leqq \sup_{x \in K} |\langle x, x'\rangle|$; in particular, if $x' \in K^0$, one has $|\langle x_0, x'\rangle| \leqq 1$ by definition, and therefore $x_0 \in K^{00} = K$; we thus see that $x_0 \in K \cap G$. Finally, the definition of the distance of a point to a set in a metric space shows that, given any $\epsilon > 0$, there exists $z' \in \Gamma$ such that $\sup_{x \in K} |\langle x, x_0' - z'\rangle| \leqq \langle x_0, x_0'\rangle + \epsilon$; since $x_0 \in K \cap G$, $\langle x_0, x_0'\rangle < 1$ by assumption; if ϵ is taken such that $\langle x_0, x_0'\rangle + \epsilon < 1$, and if $y_0' = x_0' - z'$, one has $|\langle x, y_0'\rangle| < 1$ for every $x \in K$, which is what we set out to prove.

If $\phi(\Gamma)$ is everywhere dense in E^*/H, $d = 0$ and the linear functional f_0 does not exist any more;[2] but then, one can repeat the preceding argument for $x_0 = 0$, and this completes the proof.

[2] I am indebted to the referee for calling may attention to this possibility, and to the modification the proof requires in that case.

THEOREM 3. *T is open for the bounded weak topology.*

We shall prove that a fundamental system of neighborhoods of the origin in E^* for the bounded weak topology consists of the sets K^0, where K is an arbitrary strongly compact subset of E^{**}. Once this result is secured, the argument in Theorem 2 may be applied without change, and proves Theorem 3.

To prove our assertion, consider E^* as a subspace of E^{***}; we have only to show that the bounded weak topology on E^* is induced by the bounded weak* topology on E^{***}, by Theorem 1. This will be proved if we show that any closed set F in E^* for the bounded weak topology is the intersection of E^* with a set closed in E^{***} for the bounded weak* topology and conversely that any such intersection is closed for the bounded weak topology. This second point is obvious, since closed sets in E^{***} for the bounded weak* topology are defined as sets such that their intersection with any ball S_r: $\|x'\| \leqq r$ is weakly* closed; similarly, closed sets in E^* for the bounded weak topology are defined as sets such that their intersection with any ball $S_r \cap E^*$ is weakly closed, and the weak topology on $S_r \cap E^*$ is induced by the weak* topology on S_r. To prove the first point, let F_r be the closure in S_r of $F \cap S_r$ for the weak* topology; then $F \cap S_r = F_r \cap (S_r \cap E^*)$, since by assumption $F \cap S_r$ is closed for the weak topology on E^*. Moreover, one has $F_{r'} \cap S_r \cap E^* = F_r \cap S_r \cap E^*$ for every $r' > r$; therefore, if G is the union of all F_r, $G \cap S_r \cap E^* = F \cap S_r$ for every $r > 0$, which proves that $F = G \cap E^*$. But as every S_r is weakly* closed in E^{***}, $G \cap S_r = F_{r'} \cap S_r$ for every $r' > r$, and therefore $G \cap S_r$ is weakly* closed, which shows that G is closed for the bounded weak* topology, and completes the proof.

At the end of his paper, Munroe raises the question of the closure of $T(\Delta)$ for the norm or weak topologies on E^*, when Δ is a closed (or weakly* closed) linear subspace of E^*; this, as he shows at once, is equivalent to deciding whether the sum $\Delta + \Gamma$ is or is not a closed linear subspace for these topologies. It is well known [6, p. 174] that such is *not* the case when E is any infinite-dimensional reflexive space. This shows at the same time that statement 1 in Munroe's Theorem 3.5 is erroneous. The error in the proof comes from the tacit assumption that the sum of the unit balls in Δ and Γ is still a neighborhood of O in $\Delta + \Gamma$, which need not be the case.

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UNIVERSITY OF NANCY

A PROOF THAT THE GROUP OF ALL HOMEOMORPHISMS OF THE PLANE ONTO ITSELF IS LOCALLY ARCWISE CONNECTED

M. K. FORT, JR.

Let P be a plane and let $H(P)$ be the group of all homeomorphisms of the plane P onto itself. We topologize $H(P)$ by defining convergence to mean uniform convergence on each compact subset of P. The resulting topology is equivalent to the compact-open topology defined in [1][1] by Fox. It is also known (see [4]) that $H(P)$ is a topological group under this topology. The result obtained in this paper is the following theorem.

THEOREM. *$H(P)$ is locally arcwise connected.*

1. **A metric for $H(P)$.** We assume a rectangular coordinate system for P and let d be the corresponding metric for P. For each positive number r we define $S(r)$ to be the set of all points (u, v) in P such that $\max(|u|, |v|) \leqq r$. If f and g are members of $H(P)$ we define

$$\rho(f, g) = \sup_{r>0} \min\left(1/r, \sup_{x \in S(r)} d(f(x), g(x))\right).$$

It is a routine matter to verify that ρ is a distance function which defines an admissible metric for $H(P)$. A metric which is essentially the same as ρ is used by M. Bebutoff in [2]. We shall make use of the fact that $\rho(f, g) < \epsilon$ if and only if $d(f(x), g(x)) < \epsilon$ for $\cdot \text{ll } x$ in $S(1/\epsilon)$.

2. **Isotopy and arcs.** By an isotopy we shall mean a homotopy

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

F such that $F_t \in H(P)$ whenever $0 \leqq t \leqq 1$. It follows from a theorem of Fox in [1] that an isotopy is equivalent to a continuous function on the unit interval into $H(P)$. Since the image of the unit interval under a continuous function is always arcwise connected, we see that members f and g of $H(P)$ can be joined by an arc in a subset U of $H(P)$ if and only if there exists an isotopy F such that $F_0 = f$, $F_1 = g$, and $F_t \in U$ whenever $0 \leqq t \leqq 1$.

3. Proof of the theorem. Since $H(P)$ is a topological group, it is sufficient to prove that $H(P)$ is locally arcwise connected at the identity homeomorphism I. To do this it is sufficient to show that corresponding to each positive number ϵ there is a positive number δ such that if $\rho(f, I) < \delta$, where $f \in H(P)$, then there exists an isotopy B such that $B_0 = f$, $B_1 = I$, and $\rho(B_t, I) < \epsilon$ if $0 \leqq t \leqq 1$.

Suppose $\epsilon > 0$. We may assume without loss of generality that $\epsilon < 1$. Choose a positive number δ such that $\delta < \epsilon/7$ and such that $2/\delta = (2n+2)\delta$ for some positive integer n. It is well known that $H(P)$ contains exactly two components, the component containing I consisting of all orientation preserving homeomorphisms. In this connection see [8]. We shall also assume δ small enough so that the δ-neighborhood of I contains only orientation preserving homeomorphisms. Now choose any f such that $f \in H(P)$ and $\rho(f, I) < \delta$.

We define T to be the isotopy for which T_t, $0 \leqq t \leqq 1$, is the translation which increases the first coordinate of each point of P by $t(2/\delta + \delta)$ and leaves the second coordinate invariant. Define C_t, $0 \leqq t \leqq 1$, to be the set $T_t(S(1/\epsilon))$, and then define K to be the set $U_{0 \leqq t \leqq 1} C_t$. Finally, we define $R_0 = S(1/\delta)$ and $R_1 = T_1(R_0)$.

It will be convenient to prove two lemmas.

LEMMA 1. *There exists $h \in H(P)$ such that $h \mid C_0 = f \mid C_0$, $h \mid C_1 = I \mid C_1$ and $d(x, h(x)) < \epsilon$ for all x in K.*

PROOF. Let the segment $\overline{ab}$ be the side of the square R_1 consisting of all points of R_1 whose first coordinate is a minimum and let $\overline{\alpha\beta}$ be the side of R_0 consisting of all points of R_0 whose first coordinate is a maximum. We may assume that a and α have the same second coordinate. Define x_0 to be the point on $\overline{ab}$ which is at distance δ from a. If x_i, $0 \leqq i < n$, has been defined, we then define x_{i+1} to be the point on $\overline{ab}$ which is at distance 2δ from x_i and which is between x_i and b. Since $\overline{ab}$ is of length $2/\delta$ and $2/\delta = (2n+2)\delta$, x_n is at distance δ from b.

We define y_i, $0 \leqq i \leqq n$, to be the point of $f(R_0)$ nearest to x_i which has the same second coordinate as x_i. Now define z_i, $0 \leqq i \leqq n$, to be

the point $f^{-1}(y_i)$. Since f transforms each point of R_0 a distance less than δ, it is easy to see that the points z_i all lie on $\overline{\alpha\beta}$ and moreover lie in the same order on $\overline{\alpha\beta}$ as the corresponding points x_i lie on $\overline{ab}$.

We now define h^* to be the homeomorphism which agrees with f on R_0, which agrees with I on R_1, and which transforms each segment $\overline{z_ix_i}$, $0 \leqq i \leqq n$, linearly into the segment $\overline{y_ix_i}$. We now make use of a theorem of Gehman (see [3]) and extend h^* to a homeomorphism h of P onto P.

Let Q_i, $0 \leqq i < n$, be the trapezoid (with interior) which has vertices x_i, x_{i+1}, z_{i+1}, z_i. It is easily seen that for each i, $0 \leqq i < n$, there is a rectangle with sides of length 6δ and 2δ which contains both Q_i and $h(Q_i)$. Thus, if $x \in Q_i$ we obtain $d(x, h(x)) \leqq \delta(40)^{1/2} < 7\delta = \epsilon$. It follows that if $x \in R_0 \cup R_1 \cup \cup_{i=0}^{n-1} Q_i$ then $d(x, h(x)) < \epsilon$. Moreover, it is easily seen that $K \subset R_0 \cup R_1 \cup \cup_{i=0}^{n-1} Q_i$. We therefore see that the homeomorphism h has the desired properties.

LEMMA 2. *There exists an isotopy G such that $G_0 = f$, $G_1 | C_0 = I | C_0$ and $\rho(G_t, I) < \epsilon$ whenever $0 \leqq t \leqq 1$.*

PROOF. Let h be as in Lemma 1. We then define $G_t = T_t^{-1} h T_t h^{-1} f$ whenever $0 \leqq t \leqq 1$. This clearly defines an isotopy G. Since $T_0 = I$, we readily obtain $G_0 = f$.

Let $x \in C_0$. We obtain $h^{-1}f(x) = x$ since $h | C_0 = f | C_0$. Now, using the fact that $T_1(x) \in C_1$ and $h | C_1 = I | C_1$, we obtain $hT_1h^{-1}f(x) = hT_1(x) = T_1(x)$. Therefore

$$G_1(x) = T_1^{-1} h T_1 h^{-1} f(x) = T_1^{-1} T_1(x) = x.$$

We have shown that $G_1 | C_0 = I | C_0$.

Suppose $0 \leqq t \leqq 1$ and $x \in C_0$. Then $T_t h^{-1} f(x) = T_t(x) \in K$. Thus $hT_th^{-1}f(x)$ is within ϵ of $T_t(x)$. Since T_t is a translation, this fact implies that $T_t^{-1} h T_t h^{-1} f(x)$ is within ϵ of $T_t^{-1} T_t(x) = x$. Thus $d(G_t(x), x) < \epsilon$ for each $x \in C_0 = S(1/\epsilon)$ whenever $0 \leqq t \leqq 1$. We therefore obtain $\rho(G_t, I) < \epsilon$ whenever $0 \leqq t \leqq 1$.

We now use the isotopy G defined in Lemma 2 to define the isotopy B. If $x \in P$ we define:

$$B_t(x) = G_{2t}(x) \qquad \text{for } 0 \leqq t \leqq 1/2;$$

$$B_t(x) = G_1(2(1-t)x)/2(1-t) \qquad \text{for } 1/2 < t < 1;$$

and

$$B_1(x) = I(x).$$

It is easily verified that B has the desired properties.

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UNIVERSITY OF VIRGINIA and
UNIVERSITY OF ILLINOIS

PROOF OF A CONJECTURE OF I. J. SCHOENBERG[1]

I. I. HIRSCHMAN, JR.[2]

Consider the class A of those functions $f(t)$ such that

$$(1) \qquad \begin{aligned} &f(t) \in C^{\infty} && (-\infty < t < \infty), \\ &f^{(n)}(\pm\infty) = 0 && (n = 0, 1, \cdots). \end{aligned}$$

By Rolle's theorem, if $f(t)$ belongs to A and is not identically zero, then $f^{(n)}(t)$ must have at least n changes of sign. This suggests the consideration of the subclass A' consisting of those nonzero functions of A for which $f^{(n)}(t)$ has not more than n and therefore exactly n changes of sign. Such functions correspond to our intuitive idea of the concept "bell-shaped." Many interesting functions belong to the class A', in particular all Pólya frequency functions belong to A'. See[3] [1] and [2]. There exist Pólya frequency functions vanishing identically in a neighborhood of $+\infty$ or of $-\infty$, but there does not exist one vanishing identically in neighborhoods of both $+\infty$ and $-\infty$, that is, outside an interval. This led I. J. Schoenberg[4] to conjecture that there does not exist a function of class A' vanishing outside an interval.

This conjecture is also connected with the theory established by Pólya and Wiener [3], and by Szegö [4], connecting the oscillation of the derivatives of a periodic function with its analytic character. It was, for example, proved in [4] that if B is the class of those infinitely differentiable functions, with period 2π, for which

$$(2) \qquad N_k(f(t)) = O\left[\frac{k}{\log k}\right] \qquad (k \to \infty),$$

where $N_k(f(t))$ is the number of changes of sign of $f^{(k)}(t)$ in a period, then every function of class B is the restriction to the real axis of an entire function. The validity of Schoenberg's conjecture implies that

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[3] Numbers in brackets refer to the bibliography at the end of the paper.

[4] This problem was proposed in connection with the Princeton Conference of 1946–1947.

if B' is the class of those infinitely differentiable functions, with period 2π, for which

$$N_k(f(t)) \leqq k \qquad (k = 0, 2, 4, \cdots), \tag{3}$$

then every function of class B' which vanishes with all its derivatives at a point is identically zero.

We shall now proceed to give a proof of this conjecture.

THEOREM. *If*

1. $f(t) \in C^\infty$ $(-\infty < t < \infty)$,
2. $f(t) \equiv 0$ $(|t| \geqq a)$,
3. $f^{(n)}(t)$ *has at most* n *changes of sign* $(n = 0, 1, \cdots)$,

then $f(t) \equiv 0$.

We define $\int_{-a}^{a} f(t)t^i dt = M_i$ $(i = 0, 1, 2)$. It is no restriction to suppose that $M_0 > 0$, $M_1 = 0$. Indeed if $M_0 = 0$, then $f(t) \equiv 0$, and if $M_0 \geqslant 0$, $M_1 \neq 0$, we may consider instead of $f(t)$ the function $g(t) = \epsilon f(t+b)$, where $b = M_1/M_0$, and ϵ is ± 1 so chosen that $0 \leqq g(t)$.

Since $f^{(n)}(t)$ has n changes of sign we can find n points, $z_1^{(n)}, \cdots, z_n^{(n)}$ between $-a$ and a such that

$$f^{(n)}(t) \prod_{i=1}^{n} (t - z_i^{(n)}) \tag{4}$$

is of constant sign. We define

$$s_n = -\sum_i z_i^{(n)}, \qquad r_n = \sum_{i<j} z_i^{(n)} z_j^{(n)}.$$

Integrating by parts n times one may show that

$$\int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) dt = (-1)^n n! M_0. \tag{5}$$

Using the fact that the function (4) is of constant sign, we see that

$$\left| \int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t dt \right| \leqq a n! M_0. \tag{6}$$

Now

$$\begin{aligned} \int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t dt &= (-1)^n [(n+1)! M_1 + n! s_n M_0], \\ &= (-1)^n n! s_n M_0. \end{aligned} \tag{7}$$

Combining (6) and (7), we have

$$|s_n| \leqq a. \tag{8}$$

Similarly

$$\left| \int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t^2 dt \right| \leqq a^2 n! M_0, \tag{9}$$

and

$$\int_{-a}^{a} f^{(n)}(t) \prod_i (t - z_i^{(n)}) t^2 dt = (-1)^n \left[\frac{1}{2} (n+2)! M_2 + n! r_n M_0 \right]. \tag{10}$$

Thus

$$M_2 \leqq \frac{2a^2 M_0}{(n+1)(n+2)} + \frac{2|r_n| M_0}{(n+1)(n+2)}. \tag{11}$$

Now

$$r_n = [s_n^2 - \sum (z_i^{(n)})^2]/2$$

and hence

$$|r_n| \leqq (n/2)a^2. \tag{12}$$

It follows that

$$M_2 \leqq \frac{a^2 M_0}{(n+1)}. \tag{13}$$

Since n may be taken arbitrarily large, this implies that $M_2 = 0$ and thus that $f(t) \equiv 0$.

The function which is $\exp[1/(x_2 - 1)]$ for $|x| < 1$ and 0 for $|x| \geqq 1$ is infinitely differentiable and its nth derivative has not more than $3n-2$ changes of sign for $n = 1, 2, \cdots$. Thus our theorem is no longer true if n is replaced by $3n$ in assumption 3. This example is due to D. V. Widder.

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Harvard University

ON THE "COLLECTIVE HAUSDORFF METHOD"

W. H. J. FUCHS

A sequence $\{s_n\}$ is said to be summable to S by the Hausdorff method $T\sim\mu_n$, if $\lim_{n\to\infty} t_n = S$, where $\{t_n\}$ is given by

$$\Delta^n t_0 = \sum_{k=0}^{n} (-1)^k C_{n,k} t_k = \mu_n \Delta^n s_0 \qquad (n = 0, 1, 2, \cdots). \tag{1}$$

Hausdorff methods have been extensively investigated by F. Hausdorff and others [1, 3, 5].[1]

A method of summation A is said to be *stronger* than another method B if every sequence summed by B is also summed by A. Two methods of summation are *consistent* if any sequence which can be summed by both methods has the same limit assigned to it by both of them. I shall say that A *contains* B if A is stronger than B and also consistent with B. A method containing ordinary convergence is called *regular*. F. Hausdorff proved that any two regular Hausdorff methods are consistent [3]. This enabled R. P. Agnew to introduce the "collective Hausdorff method" $\mathfrak{H}$ by the definition: $\{s_n\}$ is summable $\mathfrak{H}$ to the sum S if $\{s_n\}$ is summable to S by any regular Hausdorff method [1]. He raised at the same time the question whether there is a matrix method of summation containing $\mathfrak{H}$. I shall now show that the answer is in the negative. More precisely I prove the following:

THEOREM. *There is no matrix* $A = (a(m, n))$ *such that*

(α) $t_m = \sum_{n=0}^{\infty} a(m, n) s_n$ $(m = 0, 1, 2, \cdots)$ *converges whenever* $\{s_n\}$ *is summable* $\mathfrak{H}$ *and*

(β) *if* $\{s_n\}$ *is summed to* S *by* $\mathfrak{H}$, *then* $t_m \to S$ *as* $m \to \infty$.[2]

To start with I list a few special sequences summed by $\mathfrak{H}$.

(a) Since ordinary convergence is the Hausdorff method $C\sim 1$, $\mathfrak{H}$ sums every convergent sequence to its ordinary limit. In other words, $\mathfrak{H}$ is regular.

(b) The sequence $s_n = (-c)^n$ $(c>1)$ is summed to 0 by a suitable Euler method $E_p\sim p^{-n}$ $(p>1)$.

(c) The sequence $s_n = C_{n,k}$ is summed to 0 by Mercer's method $M_k\sim(k-n)/k(n+1)$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] To avoid double subscripts I write $a(m, n)$ instead of the usual a_{mn}.

Euler methods have been extensively studied by Knopp [4]. The special result stated as (b) can easily be verified byt he use of (1). Mercer's methods are investigated in [5]; again (c) can be verified directly by means of (1).

Suppose now that A is a matrix satisfying the conditions (α) and (β) of the theorem. By (a) this matrix defines a regular method of summation and hence, by a well known theorem [6],

$$a(m, n) \to 0 \quad \text{as} \quad m \to \infty \qquad (n = 0, 1, \cdots) \tag{2}$$

and

$$\sum_{n=0}^{\infty} a(m, n) \to 1 \quad \text{as} \quad m \to \infty .$$

By (b) and condition (α) of the theorem

$$\sum_{n=0}^{\infty} |\, a(m, n)\,| \, c^n < \infty \tag{3}$$

for every c, since $\sum a(m, n) z^n$ is an integral function.

By (3) there is an integer $n(k)$ such that

$$\sum_{l \geqq n(k)} |\, a(k, l)\,| \, 4^l < 2^{-k}. \tag{4}$$

I show next that there is a matrix B satisfying the following conditions:

(i) $b(k, l) = 0$ except perhaps for $2 \leqq l < n(k)$.

(ii) Every sequence summed by A and satisfying $|s_n| \leqq 4^n$ for all large n is summed to the same sum by B.

It is an immediate consequence of (2) and (4) that the matrix $B = (b(k, l))$ with

$$\begin{aligned} b(k, l) &= a(k, l) && (\text{for } 2 \leqq l < n(k)) \\ &= 0 && (\text{otherwise}) \end{aligned}$$

satisfies the requirements (i) and (ii). In particular B defines a regular method of summation and therefore

$$\sum_{l=0}^{\infty} b(k, l) = \sum_{l=2}^{\infty} b(k, l) \to 1 \tag{5}$$

as $k \to \infty$.

If we write

$$\Delta^n s_0 = \sigma_n,$$

$$\sum b(k, l)s_l = \sum c(k, h)\sigma_h,$$

then

$$s_n = \Delta^n \sigma_0,$$

$$b(k, l) = (-1)^l \sum_{h \geq l} c(k, h) C_{h,l}, \tag{6}$$

$$c(k, h) = (-1)^h \sum_{l \geq h} b(k, l) C_{l,h}. \tag{7}$$

By condition (i) imposed on B

$$c(k, h) = 0 \qquad (h \geqq n(k)). \tag{8}$$

By condition (ii), (c), and (7)

$$c(k, h) \to 0 \qquad \text{as } k \to \infty. \tag{9}$$

I show now that for each $k > K$, where K is a sufficiently large number, there is at least one index h such that

$$|c(k, h)| > 3^{-h}. \tag{10}$$

For otherwise, by (6),

$$|b(k, l)| \leqq \sum_{h \geq l} C_{h,l} 3^{-h} = 3^{-l}\left(1 - \frac{1}{3}\right)^{-l-1} = 3 \cdot 2^{-l-1}$$

and so

$$\left|\sum_{l=2}^{\infty} b(k, l)\right| \leqq \sum_{l=2}^{\infty} 3 \cdot 2^{-l-1} = \frac{3}{4}$$

in contradiction to (5).

By (10) we can choose $k_0 > K$, and h_0 such that $|c(k_0, h_0)| > 3^{-h_0}$. Because of (8) the element in the k_0th row of the matrix $(c(k, h))$ are 0 for $h \geqq m_0$, say. Because of (9) we can find an index $k_1 > K$ such that

$$\sum_{h < m_0} |c(k_1, h)| 3^h < \frac{1}{2}. \tag{11}$$

Also, by (10), there is an index h_1 such that $|c(k_1, h_1)| > 3^{-h_1}$. By (11) we must have $h_1 \geqq m_0$. Again $c(k_1, h) = 0$ for $h \geqq m_1$, say. Proceeding in this way we find successively $k_2, h_2, m_2, k_3, h_3, m_3, \cdots$ such that

$$\sum_{h < m_{j-1}} |c(k_j, h)| 3^h < \frac{1}{2}, \tag{12}$$

$$|c(k_j, h_j)| > 3^{-h_j}, \tag{13}$$

$$h_j \geqq m_{j-1}, \tag{14}$$

$$c(k_j, h) = 0 \qquad (h \geqq m_j). \tag{15}$$

There is always a least possible choice of m_j, but we may choose a larger number, if desired, so that there is no loss of generality in supposing

$$\sum 1/m_j < \infty$$

which implies, by (14),

$$\sum 1/h_j < \infty. \tag{16}$$

We define now a sequence $\{s_n\}$ by

$$\Delta^n s_0 = \sigma_n = 0 \qquad (n \neq h_j), \tag{17}$$

$$\sum_l b(k_j, l)s_l = \sum_h c(k_j, h)\sigma_h = (-1)^j \frac{1}{2} \quad (j = 0, 1, 2, \cdots). \tag{18}$$

By (17), (14), and (15),

$$(-1)^j \frac{1}{2} = \sum_h c(k_j, h)\sigma_h = \sum_{i \leqq j} c(k_j, h_i)\tau_i \qquad (j = 0, 1, 2, \cdots),$$

where τ_i is written for σ with the index h_i. These equations can be solved successively for $\tau_0, \tau_1, \tau_2, \cdots$ and it is easily proved by means of (12) and (13) that $|\tau_i| < 3^{h_i}$. Together with (17) this shows that $|\sigma_h| < 3^h$ and, therefore,

$$|s_n| = |\Delta^n \sigma_0| \leqq \sum C_{n,h}\sigma_h < \sum C_{n,h}3^h = 4^n.$$

It is obvious from (18) that $\{s_n\}$ is not summed by B, and so, by condition (ii) imposed on B, $\{s_n\}$ is not summed by A. On the other hand it follows from (1) and (17) that $\{s_n\}$ is summed to 0 by any Hausdorff method for which

$$\mu_n = 0 \qquad (n = h_0, h_1, h_2, \cdots). \tag{19}$$

To complete the proof of the theorem it is necessary only to show that there is a regular Hausdorff method satisfying (19).

By a well known theorem [3] the method $T \sim \mu_n$ is regular if

$$\mu_n = T(n) = \int_0^1 t^n d\phi(t),$$

where $\phi(t)$ is of bounded variation in $\langle 0, 1 \rangle$,

$$\mu_0 = \phi(1) - \phi(0) = 1,$$

and

$$\phi(0) = \phi(0+).$$

It is an easy consequence of Mellin's formula that the function[3]

$$T_1(z) = (1+z)^{-2}\prod_{j=0}^{\infty}\frac{h_j - z}{h_j + z}$$

can be written in the form $T_1(z) = \int_0^1 t^z d\phi(t)$ where $\phi(t)$ satisfies all the conditions just stated (details of the proof of this statement can be found in [2]). This proves the existence of the regular method $T_1 \sim \mu_n = T_1(n)$ satisfying (19) and completes the proof of the theorem.

This proof makes essential use of "unreasonable" Hausdorff methods for which the moment function $T(z)$ has zeros in the right half-plane. It remains an open problem whether there is a matrix method containing all reasonable Hausdorff methods, for which $T(z) \neq 0$ in $\Re z \geqq 0$.

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University of Liverpool

[3] The infinite product converges because of (16).

ON REARRANGEMENTS OF SERIES

H. M. SENGUPTA

In an interesting paper published in the Bulletin of the American Mathematical Society, R. P. Agnew[1] considers some questions on rearrangement of conditionally convergent series.

He considers the metric space E in which a point x is a permutation $(x_1, x_2, x_3, \cdots)$ of the positive integers and in which the distance $\rho(x, y)$ between two points $x \equiv (x_1, x_2, x_3, \cdots)$ and $y \equiv (y_1, y_2, y_3, \cdots)$ of E is given by the Fréchet formula

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Agnew proves that the space E is of the second category at each of its points.

He also considers the following problem: Let $C_1 + C_2 + C_3 + \cdots$ be a conditionally convergent series of real terms. Denote C_n by $C(n)$. To each $x \in E$ there corresponds a rearrangement $\sum_{n=1}^{\infty} C(x_n)$ of the series $\sum C(n)$ or $\sum C_n$ and also to each rearrangement of the series corresponds a point $x \in E$. Thus the ways in which the series may be rearranged form a set which has the potency of E. It is well known that $x \in E$ exists for which $\sum_n C(x_n)$ converges to any preassigned number, diverges to $+\infty$ or to $-\infty$, or oscillates with prescribed upper and lower limits. The set A of $x \in E$ for which $\sum_{n=1}^{\infty} C(x_n)$ converges is therefore a proper subset of E. Professor Agnew considers the nature of the set A and proves that it is of the first category so that the complementary set $E - A$ is of the second category. In point of fact he proves more than that. He shows that the set of points $x \in E$ for which $\sum_{n=1}^{\infty} C(x_n)$ has unilaterally bounded partial sums is of the first category. His first theorem runs as follows:

For each $x \in E$, except those belonging to a set of the first category,

$$\liminf_{N \to \infty} \sum_{n=1}^{N} C(x_n) = -\infty, \qquad \limsup_{N \to \infty} \sum_{n=1}^{N} C(x_n) = +\infty.$$

It is possible to add something more to the above in regard to the nature of the set A. In fact we can prove the following theorem:

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[1] Ralph Palmer Agnew, *On rearrangement of series*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 797–799.

THEOREM 1. *For each $x \in E$, except those of a set B (F_σ in E) that is an outer-limiting set of sets closed in E, we have*

$$\liminf_{N\to\infty} \sum_{n=1}^{N} C(x_n) = -\infty, \qquad \limsup_{N\to\infty} \sum_{n=1}^{N} C(x_n) = +\infty.$$

Combining the above with the result given in Agnew's paper, we may assert that the set of points x for which

$$\liminf_{N\to\infty} \sum_{n=1}^{N} C(x_n) = -\infty \quad \text{and} \quad \limsup_{N\to\infty} \sum_{n=1}^{N} C(x_n) = +\infty$$

is the inner limiting set of a sequence of everywhere dense sets open in E.

PROOF. We require the following ideas for the proof of our theorem. Consider the series $\sum_{n=1}^{\infty} C(x_n)$. Let $s_1, s_2, s_3, \cdots, s_N, \cdots$, where $s_N = \sum_{n=1}^{N} C(x_n)$, be the partial sums.

Let us denote the maximum of the numbers $(s_1, s_2, \cdots, s_N)$ by U_N. Then $s_1, U_2, U_3, \cdots$ is a monotone nondecreasing sequence of real numbers, and $\lim_{N\to\infty} U_N$ exists either as a finite number or as $+\infty$. We write $\lim_{N\to\infty} U_N$ as $\lim_{N\to\infty}$ upper bound $\sum_{n=1}^{N} C(x_n)$.

In a similar manner if u_N denotes min $(s_1, s_2, \cdots, s_N)$, then $s_1, u_2, u_3, \cdots$ is a monotone nonincreasing sequence of real numbers and $\lim_{N\to\infty} u_N$ exists either as a finite number or as $-\infty$. We denote $\lim_{N\to\infty} U_N$ by the symbol $\lim_{N\to\infty}$ lower bound $\sum_{n=1}^{N} C(x_n)$.

Now, let $h>0$ be large at pleasure. We define B_h to be the set of points of E for which $\lim_{N\to\infty}$ upper bound $\sum_{n=1}^{N} C(x_n) \leqq h$.

Now let $\xi_1, \xi_2, \xi_3, \cdots$ be a convergent sequence of points of E all belonging to B_h. Let their limit point ξ also belong to E. We shall prove that ξ also belongs to B_h.

Let $\xi_n \equiv (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \cdots)$ and $\xi \equiv (x_1, x_2, x_3, \cdots)$. The number complexes $(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \cdots)$ for $n=1, 2, 3, \cdots$ and also the complex $(x_1, x_2, x_3, \cdots)$ are different permutations of the positive integers.

If now ξ does not belong to B_h we must have

$$\lim_{N\to\infty} \text{upper bound} \sum_{n=1}^{N} C(x_n) > h.$$

So, for sufficiently large N, say $N \geqq M'$ where M' is some positive integer depending on ξ and h, we must have $U_N > h$.

In particular, therefore, $U_{M'} > h$. But $U_{M'}$ is the maximum of $(s_1, s_2, s_3, \cdots, s_{M'})$, so there is a positive integer M $(1 \leqq M \leqq M')$ such that $s_M > h$, that is, there is a positive integer M for which

$$S_M = C(x_1) + C(x_2) + \cdots + C(x_M) > h.$$

We may now take a sphere of sufficiently small radius to ensure that the first M elements of all points of E that lie in the interior of the sphere are identical with those of ξ in value and in order. So for all these points the corresponding rearranged series are such that the Mth partial sum of each of them equals $S_M = C(x_1) + C(x_2) + \cdots + C(x_M)$ and so exceeds h in value.

Now, since $\lim \xi_n = \xi$, for all sufficiently large n, say $n \geqq P$ where P is a suitable positive integer, ξ_n lies within the sphere referred to above. So

$$\sum_{r=1}^{M} C(x_r^{(n)}) = C(x_1) + C(x_2) + \cdots + C(x_M) > h$$

for $n = P, P+1, P+2, \cdots$. Therefore

$$\lim_{N\to\infty} \text{upper bound} \sum_{r=1}^{N} C(x_r^{(n)}) > h \qquad (n = P, P+1, \cdots),$$

that is, $\xi_p, \xi_{p+1}, \xi_{p+2}, \cdots$ all belong to $E - B_h$ which is contrary to the hypothesis. So ξ belongs to B_h, that is, the set B_h is closed in E.

In a like manner, we may prove that if B_{-h} stands for the set of points of E for which

$$\lim_{N\to\infty} \text{lower bound} \sum_{r=1}^{N} C(x_r) \geqq -h$$

and if $\{\zeta_1, \zeta_2, \zeta_3, \cdots\}$ is a convergent sequence of points of E all belonging to B_{-h} with ζ as the limit which itself belongs to E, the $\zeta \in B_{-h}$. So, B_{-h} is closed in E.

It follows therefore that for any positive integer n, the set $B_n + B_{-n}$ is closed in E, that is, the set of points of E, for which

$$\lim_{N\to\infty} \text{upper bound} \sum_{r=1}^{N} C(x_r) \leqq n$$

or

$$\lim_{N\to\infty} \text{lower bound} \sum_{r=1}^{N} C(x_r) \geqq -n$$

is closed in E.

The set B referred to above is, therefore, given by

$$B = \sum_{n=1}^{\infty} (B_n + B_{-n}).$$

So, B is a set F_σ in E, that is, the outer limiting set of a sequence of sets closed in E.

It follows that the set of points $x \in E$ for which

$$\liminf_{N\to\infty} \sum_{n=1}^{N} C(x_n) = -\infty \quad \text{and} \quad \limsup_{N\to\infty} \sum_{n=1}^{N} C(x_n) = +\infty$$

is G_δ in E.

We next propose to derive some simple properties about the power of E.

THEOREM 2. *The set E has the power c of the continuum.*

PROOF. Let p be the power of the set E. Let us associate the point $x=(x_1, x_2, x_3, \cdots)$ of the set E to the irrational number α which is the value of the simple continued fraction

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots .$$

Thus to each point $x \in E$ there corresponds an irrational number α in $0<\alpha<1$ and to two distinct points x and y of E correspond two distinct irrational numbers in the above range. Thus the set E is equivalent to a proper subset of the real numbers. Therefore

$$p \leqq C.$$

On the other hand, to any real number β $(0<\beta<1)$ there corresponds a rearrangement of the conditionally convergent series $\sum_{n=1}^{\infty} C_n$ such that the rearranged series converges to β. In fact there are an infinity of such rearrangements with the sum β. So to any real number correspond an infinity of distinct points of E. Again to any other real number γ, there correspond an infinity of other distinct points of E, each of which is different from any of those corresponding to β. Thus the set of real numbers is equivalent to a proper subset of the aggregate of points of E. Therefore $p \geqq C$. Combining these two, we have $p=c$.

THEOREM 3. *Every point of the metric space E is of degree c (power of the continuum) in E.*[2]

PROOF. Take any point $x=(x_1, x_2, x_3, \cdots)$ of E. Then

$$C(x_1) + C(x_2) + C(x_3) + \cdots$$

is a rearrangement of the given conditionally convergent series. Let N be a positive integer. Let $0<\alpha<1$ and let

[2] The proof of Theorem 3 is due to the referee. It may be remarked that a proof of the above modelled on the plan of proof of Theorem 2 may also be easily given.

$$\alpha = .\alpha_1\alpha_2\alpha_3\alpha_4 \cdots$$

be the dyadic expansion of α which may terminate with zeros but not with ones.

Let $x_n^{(\alpha)} = x_n$ $(n = 1, 2, 3, \cdots, N+1)$ and if $\alpha_p = 0$, let

$$x_{N+2p}^{(\alpha)} = x_{N+2p}, \qquad x_{N+2p+1}^{(\alpha)} = x_{N+2p+1}$$

and if $\alpha_p = 1$, let

$$x_{N+2p}^{(\alpha)} = x_{N+2p+1}, \qquad x_{N+2p+1}^{(\alpha)} = x_{N+2p}.$$

Then to different numbers α correspond different points

$$x^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)}, \cdots).$$

All points $x^{(\alpha)}$ lie within the sphere with center x and radius 2^{-N}. And since $\lim C(x_n) = 0$, the series

$$C(x_1^{(\alpha)}) + C(x_2^{(\alpha)}) + C(x_3^{(\alpha)}) + \cdots$$

has, for each α, exactly the same limits of oscillation as the series $\sum C(x_n)$.

This proves that the points of E that lie within the sphere $S(x, 2^{-N})$ has the power of the continuum.

It proves more. It easily leads to the theorem:

THEOREM 4. *Every point of E is the limit point of a set of points of E of power c at which the rearrangements of the conditionally convergent series behave in any prescribed manner.*

PROOF. Take any point $x = (x_1, x_2, x_3, \cdots)$. Let l and L be any two real numbers, $-\infty$ and $+\infty$ not excepted, such that $-\infty \leqq l \leqq L \leqq +\infty$. In a sphere with x as center and 2^{-N} as radius there exists a point $X = (x_1, x_2, \cdots, x_{N+1}, X_{N+2}, X_{N+3}, \cdots)$ at which the rearranged series has l and L as its limits of oscillation. Again, by the above, in a sufficiently small sphere with X as center lying entirely within the first sphere, an infinite set of points of the power C exist at which the rearranged series has the same limits of oscillation l and L.

Thus in every neighbourhood of x, there exists a set of points of the power of the continuum, at which the rearranged series has any prescribed limits of oscillation.

DACCA, INDIA

A THEOREM ON THE ACCESSIBILITY OF BOUNDARY PARTS OF AN OPEN POINT SET

EBERHARD HOPF

Let Ω be a bounded open set in ordinary n-space. Denote by p a point of that space. A closed subset of the boundary of Ω is called accessible from within Ω if Ω contains a semi-open Jordan curve $p(t)$, $0 \leqq t < \infty$, whose points of accumulation, for $t \to \infty$, precisely constitute that boundary part. In this note we prove the following theorem.

THEOREM 1. *If Ω is a bounded open set and if Ω' is a connected*[1] *open subset Ω' of Ω whose boundary has at least a point in common with the boundary of Ω, then the set C of all of these common boundary points contains a set accessible from within Ω. If C is the sum of two disjoint closed sets then each of these sets contains a set accessible from within Ω.*

The theorem was established and proved in order to bridge a gap in the proof of a classical theorem on surfaces $z(x, y)$ of nonpositive Gaussian curvature. Theorem 1 might, however, be of interest in other respects. The completed proof of the differential geometric theorem is given in the subsequent paper. The special case of Theorem 1 needed for the completion is that where the boundaries of the open set Ω and of the open and connected sets $\Omega' \subset \Omega$ have only one or two points in common. None of these points need, of course, be accessible from Ω'. Theorem 1, however, ascertains that each of these points is accessible from Ω.

The conclusion of the theorem remains valid if the open set $\Omega' \subset \Omega$, instead of being connected, is merely supposed to be connected to the vicinity of C. We call an open set Ω' connected to the vicinity of a closed set S if there is a neighborhood N^* of S (open set containing S) with the following property. For every neighborhood N of S there is a Jordan arc whose interior lies in Ω' and whose end points lie in N and on the boundary of N^*, respectively. The generalized Theorem 2 is found to be more easily proved than Theorem 1. It is convenient to extend the notion of accessibility of a closed set S from an open set Ω to arbitrary closed sets (not necessarily parts of the boundary of Ω).

THEOREM 2. *Let Ω and $\Omega' \subset \Omega$ be two bounded open sets and denote by S a closed set which contains all points common to the boundaries of Ω*

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[1] Throughout this note the notion of connectedness is meant in the sense of arcwise connectedness.

and Ω'. Let Ω' be connected to the vicinity of S. Then S contains a set accessible from within Ω. If S splits into two disjoint closed sets the same is true for each part.

PROOF OF THEOREM 2. Recall the set N^* in the definition of vicinity-connectedness. Let N_1 and N_2 be two arbitrary neighborhoods of S subject to the conditions

$$N_1 \subset N^*, \text{ closure of } N_2 \subset N_1. \tag{1}$$

Let

$$N_1\Omega = \sum C_\nu \tag{2}$$

be the decomposition of the open set N_1 into disjoint connected components. The open sets

$$C_n' = C_n N_2 \Omega' \tag{3}$$

are disjoint and satisfy, according as $N_2 \subset N_1$ and $\Omega' \subset \Omega$, the relation

$$N_2\Omega' = \sum C_\nu'. \tag{4}$$

N_2 contains S and S in turn contains, according to hypothesis, boundary points of Ω'. The open set N_2 contains, therefore, points of Ω' and the sets C_n' cannot all be empty.

We first show that a boundary point of C_n is either a boundary point of Ω or a boundary point of N_1. In fact, a boundary point p of C_n is a point of accumulation of N_1. If p is not in N_1 it must lie on the boundary of N_1. Suppose that $p \subset N_1$. As a boundary point of the open set C_n, p cannot belong to C_n. Furthermore, p cannot belong to any other C_m. Otherwise a whole neighborhood of p would belong to C_m and $C_n C_m$ would not be empty since the neighborhood contains points of C_n. p therefore is not in $\sum C_\nu = N_1\Omega$ and, as $p \subset N_1$, not in Ω. On the other hand, every neighborhood of p contains points of C_n, and $C_n \subset \Omega$. p must be a boundary point of Ω.

The boundary of C_n and the boundary of the subset C_n' of C_n cannot have common points except on S. In fact, such a point must be a boundary point of Ω since, according as $C_n' \subset N_2$ and as the second inequality (1) holds, it cannot lie on the boundary of N_1. It must also be a boundary point of Ω' since it is a point of accumulation for $\sum_1^1 C_n' \subset \Omega' \subset \Omega$. According to the hypothesis, a common boundary point of Ω and Ω' can only lie on S.

The main point of the proof is to show that among the sets C_n' there is at least one that is connected to the vicinity of S. To this purpose we first consider the "special" C_n', that is, those C_n' which have a

boundary point, say p_n, on the boundary of N_2. We prove that the special C_n' must be finite in number. If there were infinitely many, the corresponding points p_n would have a point accumulation p^* on the boundary of N_2. Every sufficiently small neighborhood V of p^* would, according to the second inequality (1), contain no points of the boundary of N_1. We can always assume V to be connected. V would contain points p_n, p_m belonging to two disjoint open sets C_n, C_m and, therefore, all the points of a suitable Jordan arc joining them. This arc, and therefore V, must contain a boundary point of C_n. This point is either a boundary point of Ω or of N_1. The latter case cannot occur in V. Consequently, every neighborhood of p^* contains a boundary point of Ω, and, therefore, p^* is itself a boundary point of Ω. From $p_n \subset \Omega'$ and $\Omega' \subset \Omega$ it is obvious that p^* is also a boundary point of Ω'. This is a contradiction, since a common boundary point of Ω and Ω' must, according to the hypothesis, lie on $S \subset N_2$ and, therefore, cannot lie on the boundary of N_2. Hence the special C_n' must be finite in number. Now we make use of the hypothesis that the open set Ω' is connected to the vicinity of S. For any neighborhood N of S,

$$N \subset N_2,$$

there is a Jordan arc leading from some point of the boundary of N^* entirely through Ω' to some point of N. On this arc there is, according to (1), a last point of entry into N_2. This is the initial point of a subarc which thus joins some boundary point of N_2 entirely through $N_2\Omega' = \sum C_\nu'$ to some point of N. Since the C_ν ($C_\nu' \subset C_\nu$) are mutually disconnected, the subarc must lie wholly in one of the C_ν'. This must, obviously, be one of the finitely many special C_ν'. If we let N run through a sequence of neighborhoods of S closing down on S, and if we attach an arc and a subarc to each of these N, we see that one of the C_ν', say C_n', must contain infinitely many of these subarcs. C_n' is evidently connected to the vicinity of S.

The result obtained is this. Given any two neighborhoods N_1N_2 of S satisfying (1), there exist two open sets Ω_1 and $\Omega_1' \subset \Omega_1$,

$$\Omega_1 \subset \Omega, \qquad \Omega_1 \subset N_1, \qquad \Omega_1' \subset N_2,$$

with the following properties. The common points of the boundaries of Ω_1 and Ω_1' lie on S. Ω_1 is connected. Ω_1' is connected to the vicinity of S.[2] The three sets Ω_1, Ω_1', and S thus have precisely the properties

[2] The neighborhood from whose boundary the arcs are drawn is N_2. In the case of Ω' it was N^*, which could, of course, be replaced by N_1.

supposed about Ω, Ω', and S with the important additional information that Ω_1 is connected.

We now start from a sequence of neighborhoods N_1, N_2, N_3, $\cdots$ of S closing down on S. We choose them such that

$$(\text{closure of } N_{n+1}) \subset N_n.$$

In the same way as before, starting this time from Ω_1 and Ω_1', we can construct two open sets Ω_2 and $\Omega_2' \subset \Omega_2$,

$$\Omega_2 \subset \Omega_1, \qquad \Omega_2 \subset N_2, \qquad \Omega_2' \subset N_2,$$

such that Ω_2, Ω_2', and S have the same properties as Ω_1, Ω_1', and S.[3] Continuing in this manner we obtain a sequence of connected open sets with the properties

$$\Omega \supset \Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots, \ \Omega_n \subset N_n.$$

The Ω_n' are no longer needed. In each Ω_n we choose a point q_n. As Ω_n is connected, q_n and q_{n+1} can be joined by an arc $\alpha_n \subset \Omega_n \subset N_n$. The curve $\alpha_1+\alpha_2+\alpha_3+\cdots$ has the desired properties, except perhaps simplicity. Every point of accumulation, as $t \to \infty$, lies in the common part of the N_n, which is S since S is closed. By the well known procedure of "omitting loops" we obtain a simple Jordan curve with the same properties.

The additional concluding statement of Theorem 1 is proved in the following way. Let $C = C_1 + C_2$, where the parts are closed and disjoint. There exist neighborhoods N_1 of C_1 and N_2 of C_2 without common points. The boundaries of the sets $\Omega + N_2$ and Ω' obviously have precisely C_1 as their common part. According to Theorem 1, C_1 is therefore accessible from $\Omega + N_2$. The Jordan curve in $\Omega + N_2$ that converges towards C_1 must, from a certain point on, lie in N_1 and thus outside of N_2, in other words in Ω. C_1 is, therefore, accessible from Ω.

NEW YORK UNIVERSITY

[3] N^2 is the neighborhood from whose boundary the arcs are drawn through Ω_2 to the vicinity of S.

ON S. BERNSTEIN'S THEOREM ON SURFACES $z(x, y)$ OF NONPOSITIVE CURVATURE

EBERHARD HOPF

The following classical theorem is due to S. Bernstein: If $z=z(x, y)$ is of class C'' in the whole x-y-plane and if

$$z_{xx}z_{yy} - z_{xy}^2 \leqq 0, z_{xx}z_{yy} - z_{xy}^2 \not\equiv 0, \tag{1}$$

then $z(x, y)$ cannot be bounded.[1] The original proof was found to contain a gap of topological nature. It is the purpose of this note to bridge this gap and to prove a somewhat more general theorem.

THEOREM. *If $z(x, y)$ belongs to C'' in the whole x-y-plane and satisfies* (1) *then $z(x, y)$ cannot be $o(r)$ where r is the distance of (x, y) from an arbitrarily chosen fixed point.*

That this estimate of the order of magnitude at infinity cannot be essentially improved is shown by examples of the form $z=f(x)-g(y)$, $f''>0$, $g''>0$, where f and g can be chosen such that the order is just $O(r)$. A still open question is whether $z(x, y)$ can or cannot be $o(r)$ along a special sequence of radii $r_\nu \to \infty$.

In proving the theorem we shall, essentially, follow Bernstein's original arguments. For the sake of completeness the arguments will be repeated.

LEMMA 1 OF BERNSTEIN. *Let $z(x, y)$ be of class C'' in a bounded open set R and let $z_{xx}z_{yy}-z_{xy}^2 \leqq 0$ in R. If z is continuous on the boundary B of R and if $z \leqq 0$ on B, then $z \leqq 0$ in the whole of R.*

PROOF (according to M. Shiffman). Let C be a circle in the plane $z=0$ whose interior contains $R+B$. Consider the parts $z \geqq 0$ of all possible spheres which intersect the plane $z=0$ in C. They form a

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[1] S. Bernstein, *Ueber ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus*, Math. Zeit. vol. 26 (1927) pp. 151–158. This is the translation of a paper written in French (*Sur une théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique*) and published in 1914 by the Mathematical Society of Charkow.

Note added in proof. It had escaped the author's attention that Bernstein himself had already published stronger results, which include the generalized theorem. See his paper, *Renforcement de mon théorème sur les surfaces à courbure negative*, Bull. Acad. Sci. URSS. Sér. Math. (1942). His proof, however, contains the same gap as the original one. The argument fails if all four domains Ω spiral infinitely often around the origin.

monotonic family of surfaces $z=Z(x, y; \lambda)$ above $R+B$ with $\lambda=\max Z\geqq 0$ taken on the sphere. We have $Z=0$ for $\lambda=0$, $Z>0$ in $R+B$ for $\lambda>0$, and $Z\to\infty$ uniformly in $R+B$ for $\lambda\to\infty$. Let λ^* be the greatest lower bound of all λ for which everywhere

$$Z(x, y; \lambda) \geqq z(x, y) \qquad \text{in } R+B. \tag{2}$$

It is to be proved that $\lambda^*=0$. Suppose that $\lambda^*>0$. For reasons of continuity (2) must hold also for $\lambda=\lambda^*$ and there must be some point x^*, y^* in $R+B$ where $=$ holds in (2), $\lambda=\lambda^*$. This point must lie in R because, according to hypothesis, $z\leqq 0$ and $Z>0$ ($\lambda>0$) on B. In the space point $x^*, y^*, z^*=z(x^*, y^*)$ the surface must, according to (2), $\lambda=\lambda^*$, be internally tangent to a sphere. Evidently the point of tangency would be a point of positive Gaussian curvature for $z(x, y)$, in contradiction to the hypothesis.

LEMMA 2 (*essentially due to Bernstein*). *Let $z(x, y)$ be of class C'' and let $z_{xx}z_{yy}-z_{xy}^2\geqq 0$ in a connected open set R. Let $z>0$ in R and let z be continuous and equal to 0 on the boundary B of R. Suppose that R can be placed in an angle less than π. Consider the chord of this angle perpendicular to the line bisecting the angle and at distance u from the vertex. Then the maximum $M(u)$ of $z(x, y)$ on the $(R+B)$-part of this chord is defined in an infinite u-interval and nowhere convex in this interval,*

$$(u_3-u_1)M(u_2) \leqq (u_3-u_2)M(u_1)+(u_2-u_1)M(u_3), \qquad u_1<u_2<u_3. \tag{3}$$

COROLLARY. *Under the hypothesis of Lemma 2, there exists a positive constant c such that $M(u)>cu$ for all u sufficiently large.*

PROOF. According to Lemma 1, R must be unbounded. Since R is connected, $M(u)$ must be defined for all $u\geqq u_0$, and $M(u)\geqq 0$, $M(u_0)=0$. Suppose, now, that (3) were false for three fixed values $u_1<u_2<u_3$,

$$(u_3-u_1)M(u_2) > (u_3-u_2)M(u_1)+(u_2-u_1)M(u_3). \tag{4}$$

As the linear function of x, y

$$l(x, y) = au+b \qquad (a, b \text{ constants})$$

always satisfies the equality in (3), the chord maxima $M^*(u)=M(u)-(au+b)$ of the function

$$z^*(x, y) = z(x, y) - l(x, y) \tag{5}$$

again satisfy (4) for those fixed u_i,

(6) $(u_3 - u_1)M^*(u_2) > (u_3 - u_2)M^*(u_1) + (u_2 - u_1)M^*(u_3).$

If a, b are chosen such that

(7) $$l(u_i) = M(u_i) \geqq 0, \qquad i = 1, 3,$$

we have $M^*(u_i) = 0$ for $i = 1, 3$ and, therefore,

(8) $$z^* \leqq 0 \qquad \text{on } u = u_1 \text{ and } u = u_3.$$

(6) now implies that $M^*(u_2) > 0$ and that

(9) $$z^* > 0 \qquad \text{somewhere on } u = u_2.$$

From (7) we infer that $l \geqq 0$ for $u_1 < u < u_3$. This shows, in conjunction with the hypothesis that $z = 0$ on B, that

(10) $$z^* \leqq 0 \qquad \text{on } B \text{ for } u_1 \leqq u \leqq u_3.$$

If R' denotes the set common to R and to $u_1 < u < u_3$, (8) and (10) mean that $z^* \leqq 0$ on the entire boundary of the bounded open set R'. Since

$$z^*_{xx} z^*_{xy} - (z^*_{xy})^2 \leqq 0$$

in R', Lemma 1 shows that $z^* \leqq 0$ in the whole of R'. This contradiction to (9) proves Lemma 2.

The corollary is an obvious consequence of the properties of nowhere convex functions $M(u) \geqq 0$, $M(u_0) = 0$.

LEMMA 3. *Suppose that the hypotheses of Lemma 2 are fulfilled. Let $N(r)$ be the maximum of $z(x, y)$ on the $(R+B)$-part of the circle of radius r about some fixed point. Then there exists a positive constant c' such that $N(r) > c'r$ for all r sufficiently large.*

PROOF. For all r sufficiently large the circle about the fixed point intersects each of the two rays of the angle just once. To each such value r_1 of r belongs a number $u = u_1$ defined as the maximum value of u at which the chord lies in the area $r \leqq r_1$. It is geometrically obvious that

(11) $$u_1/r_1 \geqq c'' > 0 \qquad (c'' \text{ constant})$$

for all sufficiently large values of r_1. We further restrict r_1 to a range $r_1 \geqq a$ where a is chosen such that both the chord u_1 and the circle r_1 always intersect $R+B$. Consider the region R_1 common to R and to $r < r_1$. On the part of the boundary of R_1 with $r < r_1$ we have, according to hypothesis, $z = 0$. On $r = r_1$, $z \leqq N(r_1)$ and $N(r_1) \geqq 0$. The function $z^* = z - N(r_1)$ has, therefore, values not greater than 0 on the whole boundary of the bounded open set R_1. From Lemma 1 one in-

fers that $z^* \leqq 0$ everywhere in R_1 and, in particular, on the chord $u = u_1$ which we know to lie in $r \leqq r_1$. Therefore,

$$M(u_1) \leqq N(r_1) \tag{12}$$

holds for all r_1 sufficiently large and the lemma evidently follows from (11), (12), and from the corollary of Lemma 2.

PROOF OF THE THEOREM. The points x, y where $z = z(x, y)$ has negative curvature form an open set. Among these points is surely one where $z_x = z_y = 0$ does not hold. We may suppose that

$$z = z_x = 0, \qquad z_y = q_0 > 0,$$
$$z_{xx}z_{yy} - z_{xy}^2 < 0 \qquad \text{at } x = y = 0.$$

The function $\xi = z - q_0 y$ satisfies $\xi_{xx}\xi_{yy} - \xi_{xy}^2 \leqq 0$ and

$$\xi = \xi_x = \xi_y = 0, \qquad \xi_{xx}\xi_{yy} - \xi_{xy}^2 < 0 \qquad \text{at } x = y = 0. \tag{13}$$

Suppose, now, the theorem were false. There would exist a function $\epsilon(r)$, $\epsilon \to \infty$, such that

$$|z(x, y)| < r\epsilon(r), \tag{14}$$

where r is the distance of x, y from some fixed point. The statement is easily seen to be independent of the location of this initial point. We may suppose that $r^2 = x^2 + y^2$. We can also assume that ϵ is continuous and decreasing for $r \geqq r_0$ and that

$$\epsilon(r_0) = q_0 \qquad (r_0 > 0).$$

The region S defined by the inequalities $r < r_0$ and, for $r \geqq r_0$,

$$|y| < \frac{r}{q_0}\epsilon(r) \tag{15}$$

is bounded by two continuous curves L^+, L^-, defined respectively by

$$y = \pm \frac{r}{q_0}\epsilon(r), \qquad r \geqq r_0, \tag{16}$$

or, in polar coordinates, by $q_0 \sin \phi = \pm\epsilon(r)$. Each of these curves reaches towards $x = \pm\infty$. We also mention that $|y/r| \to 0$ along both curves as $|x| \to \infty$. It follows from (14), (15), and (16) that $\xi = z - q_0 y$ satisfies

$$\xi < 0 \text{ on } L^+, \qquad \xi > 0 \text{ on } L^-, \tag{17}$$

and

$$|\xi| < 2r\epsilon(r) \text{ in } S, \qquad r > r_0. \tag{18}$$

These properties of $\xi(x, y)$ together with the ones mentioned before are contradictory. We proceed to prove this.

(13) implies the existence of two straight line segments crossing each other in (0, 0) such that, except at (0, 0), $\xi>0$ on one and $\xi<0$ on the other segment. Two points on the same segment but on opposite sides of (0, 0) can never be joined by a Jordan arc on which ξ does not change sign. Otherwise a bounded region would be enclosed in which $\xi\neq 0$ and on whose boundary $\xi=0$, which contradicts Lemma 1. The open set where $\xi\neq 0$, therefore, contains exactly four components that contain, respectively, the four partial segments obtained by removing the point (0, 0),

$$\xi > 0 \text{ in } \Omega_i^+, \qquad \xi < 0 \text{ in } \Omega_i^-, \qquad i = 1, 2. \tag{19}$$

$\xi=0$ on the boundary of each of these four regions. Choose a $p>0$ such that each of the four connected sets Ω contains a point in which

$$|\xi| > p.$$

The set where $|\xi|>p$ must contain four different components Ω' such that each of them contains one of those four points,

$$\Omega_i'^{+} \subset \Omega_i^+, \qquad \Omega_i'^{-} \subset \Omega_i^-. \tag{20}$$

Continuity of ξ implies that these inequalities remain true if the left-hand sets are replaced by their closures. On using these auxiliary regions Ω' we shall be able to bridge the gap mentioned in the beginning.

We prove that each of the regions Ω contains a Jordan curve $x(t)$, $y(t)$ that lies in the strip S and for which $x\to\infty$ as $t\to\infty$ and $x\to-\infty$ as $t\to-\infty$. This is trivial if Ω contains a point of one of the lines L because, according to (17) and to the definition of Ω, Ω would then contain all points of L. Suppose, now, that Ω contains no point of $L^+ + L^-$. Since (0, 0) is an interior point of S and a boundary point of Ω the connected open set must be entirely contained in the strip S,

$$\Omega \subset S. \tag{21}$$

Consider the component Ω' of the set $|\xi|>p$ whose closure is, according to (20), contained in Ω. The notion of closure was hitherto understood in the sense of adding all finite points of accumulation. We now add the two infinite points $x=\pm\infty$ to the boundary of the strip S. In the new sense of closure thus involved the situation is this. Ω' is a connected open subset of Ω. The boundaries (in the extended sense) of Ω and Ω' cannot have common points except $x=-\infty$

and $x=+\infty$. We now show that they must be common boundary points of Ω and Ω'. It is sufficient to show that $x=+\infty$ is a boundary point of Ω'. If this were false, Ω' would lie in some half-plane $x<x_0$. The common part of S and of this half-plane could be placed within an angle less than π that is bisected by the x-axis (even within an angle of arbitrarily small opening). The definition of Ω' implies that one of the functions $-\xi-p$, $\xi-p$ is greater than 0 in Ω' and equal to 0 on its boundary. Lemma 3 can, therefore, be applied to this function in Ω' (which region must in view of Lemma 1 be unbounded). The statement of this lemma is, however, incompatible with the property (18) of ξ, $\epsilon\to 0$ for $r\to\infty$. $x=\infty$ is, therefore, a boundary point of Ω' (and $x=-\infty$ as well). It now follows from the topological theorem proved in the preceding paper[2] (by mapping S on a bounded set it is seen that the theorem applies to our case) that each of the two boundary points $x=\pm\infty$ is accessible from Ω, which is precisely what we wanted to prove.[3]

The statement just proved, namely that each of the four regions Ω contains a Jordan curve in S that joins the two infinite boundary points $|x|=\infty$, is, however, contradictory. Let α_i be a Jordan curve that joins (0, 0) (which is a boundary point of each Ω) through $S\Omega_i^+$ to $x=+\infty$. The "closed" curve $\alpha_1+\alpha_2$ must enclose one of the regions Ω^-. $S\Omega^-$ clearly could not contain a Jordan curve running towards $x=-\infty$, which is in contradiction to the statement proved above. This finishes the proof of the theorem.

NEW YORK UNIVERSITY

[2] E. Hopf, *A theorem on the accessibility of boundary parts of an open point set.*

[3] The error in the original proof of the theorem was, essentially, to infer the accessibility of $x=\infty$ from the mere fact that it is a boundary point of Ω.

A REMARK ON A THEOREM OF SERGE BERNSTEIN

EARL J. MICKLE

1. Introduction. In this note we shall discuss the following theorem of Bernstein (Math. Zeit. vol. 26 (1927) pp. 551–558).

THEOREM. *Let $f(x, y)$ be a real-valued function which satisfies the following conditions.*

(a) *$f(x, y)$ is of class C'' in the entire xy-plane, that is, $f(x, y)$ is defined and has continuous partial derivative of the second order in the entire xy-plane.*

(b) $f_{xx}f_{yy}-f_{xy}^2 \leqq 0$, $f_{xx}f_{yy}-f_{xy}^2 \not\equiv 0$.

Then $f(x, y)$ is not bounded.

We shall now review one of the main steps in the proof of this theorem. We can assume without loss of generality that $f(x, y)$ satisfies the further conditions:

(c) $f(0, 0)=0, f_x(0, 0)=0, q_0=f_y(0, 0)>0, f_{xx}(0, 0)f_{yy}(0, 0)-f_{xy}^2(0, 0)<0$. The proof of the above theorem depends on the following lemma.

LEMMA 1.1. *Let $f(x, y)$ be a real-valued function which satisfies the above conditions* (a), (b), (c) *and assume that there is a constant $h>0$ such that $-h<f(x, y)<h$. Then the set of points where the function $g(x, y)=f(x, y)-q_0y$ is different from zero contains a component D which lies in the strip $-h/q_0<y<h/q_0$ and is bounded on either the right or the left.*

E. Hopf noted that the proof given of this lemma does not rule out the case where each of the components which lies in the strip $-h/q_0<y<h/q_0$ might oscillate back and forth so that points arbitrarily far from the origin in both directions occur in it. Hopf (Bull. Amer. Math. Soc. vol. 56 (1950) pp. 80–85) then gives a correct proof of the above theorem. In this note we give an alternate and somewhat simpler proof. We shall show that the argument of Bernstein can be used to prove the above theorem by the use of the following lemma, the proof of which is given in §3.

LEMMA 1.2. *Let $f(x, y)$ be a real-valued function which satisfies the above conditions* (a), (b), (c) *and assume that there is a constant $h>0$ such that $-h<f(x, y)<h$. Then there exists a real-valued function $f^*(x, y)$ which satisfies the following conditions.*

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(a*) *$f^*(x, y)$ is of class C'' in the entire xy-plane.*

(b*) $f^*_{xx}f^*_{yy}-f^{*2}_{xy}\leqq 0$.

(c*) $f^*(0, 0)=f^*_x(0, 0)=0, f^*_y(0, 0)=f_y(0, 0)=q_0, f^*_{xx}(0, 0)=f_{xx}(0, 0)$, $f^*_{xy}(0, 0)=f_{xy}(0, 0), f^*_{yy}(0, 0)=f_{yy}(0, 0)$.

(d*) *There is an $\eta>0$ such that $|f(x, y)-f^*(x, y)|<\eta$.*

(e*) *The set of points where $g^*(x, y)=f^*(x, y)-q_0y$ is different from zero contains a component D^* which lies in the strip $-h/q_0<y<h/q_0$ and is bounded on either the right or the left.*

The argument of Bernstein can then be used to show that $f^*(x, y)$ can not be bounded, and hence by (*d*) that $f(x, y)$ can not be bounded.

2. **Preliminary lemma.** Let $G(x, y; \lambda)$ be a one-parameter family of real-valued functions which satisfy the following conditions for $-\eta<\lambda<\eta$, $\eta>0$.

C_1. $G(x, y; \lambda)$ is of class C'' in the entire xy-plane.

C_2. $G_{xx}G_{yy}-G^2_{xy}\leqq 0$.

C_3. $G(0, 0; \lambda)=G_x(0, 0; \lambda)=G_y(0, 0; \lambda)=0$.

C_4. $G_{xx}(0, 0; \lambda)$, $G_{xy}(0, 0; \lambda)$, $G_{yy}(0, 0; \lambda)$ are constant and $G_{xx}(0, 0; \lambda)G_{yy}(0, 0; \lambda)-G^2_{xy}(0, 0; \lambda)<0$.

C_5. There is a number $\alpha>0$ such that $G(x, \alpha; 0)<0, G(x, -\alpha; 0)>0$ for $-\infty<x<\infty$.

C_6. $G(x, y; \lambda_1)<G(x, y; \lambda_2)$ for $\lambda_1<\lambda_2$, $(x, y)\neq(0, 0)$.

Following the argument of Bernstein we have the following facts. Under the above conditions there exist two numbers k_1, k_2 depending upon $G_{xx}(0, 0; \lambda)$, $G_{xy}(0, 0; \lambda)$, $G_{yy}(0, 0; \lambda)$ (see C_4) and a number $\epsilon(\lambda)>0$ such that $G(x, y; \lambda)>0$ for $y=k_1x$, $0<|x|<\epsilon(\lambda)$ and $G(x, y; \lambda)<0$ for $y=k_2x$, $0<|x|<\epsilon(\lambda)$. Let $\gamma_1(\lambda)$ be the segment $y=k_1x$, $0<x<\epsilon(\lambda)$, $\gamma_3(\lambda)$ be the segment $y=k_1x$, $-\epsilon(\lambda)<x<0$, $\gamma_2(\lambda)$ be the segment $y=k_2x$, $0<x<\epsilon(\lambda)$, and $\gamma_4(\lambda)$ be the segment $y=k_2x$, $-\epsilon(\lambda)<x<0$. For each λ the set of points in the xy-plane where $G(x, y; \lambda)\neq 0$ contains four distinct components $D_i(\lambda)\supset\gamma_i(\lambda)$, $i=1, 2, 3, 4$. For the components $D_i(\lambda)$

$$
\begin{aligned}
&G(x, y; \lambda)>0 \text{ for } (x, y)\in D_1(\lambda) \text{ or } D_3(\lambda),\\
&G(x, y; \lambda)<0 \text{ for } (x, y)\in D_2(\lambda) \text{ or } D_4(\lambda),
\end{aligned}
\tag{1}
$$

and, by condition C_6, for the closures $c[D_i(\lambda)]$ of $D_i(\lambda)$ we have the relations

$$
\begin{aligned}
&c[D_i(\lambda_1)]-(0, 0)\subset D_i(\lambda_2), \qquad \lambda_1<\lambda_2,\ i=1, 3,\\
&c[D_i(\lambda_2)]-(0, 0)\subset D_i(\lambda_1), \qquad \lambda_1<\lambda_2,\ i=2, 4.
\end{aligned}
\tag{2}
$$

From (2) it thus follows that

$$c[D_1(\lambda)]c[D_3(\lambda)] = C[D_2(\lambda)]c[D_4(\lambda)] = (0, 0). \tag{3}$$

We now prove the following preliminary lemma.

LEMMA 2.1. *Let $G(x, y; \lambda)$ be a one-parameter family of real-valued functions which satisfy the above conditions* $C_1, \cdots, C_6$. *Then there is a number λ_0, $-\eta < \lambda_0 < \eta$, such that the set of points in the xy-plane where $G(x, y; \lambda_0) \neq 0$ contains a component which lies in the strip $-\alpha < y < \alpha$ and is bounded on either the right or the left.*

PROOF. Consider the components $D_i(\lambda)$ described above.

Case 1. One of the components $D_i(0)$, say $D_{i_0}(0)$, does not cross the line $x = x_0$. By condition C_5, $D_{i_0}(0)$ does not cross either of the lines $y = \pm\alpha$ (otherwise it would have to contain the entire line and hence would cross the line $x = x_0$). Thus $D_{i_0}(0)$ lies in the strip $-\alpha < y < \alpha$ and is bounded on either the right or the left. Take $\lambda_0 = 0$ in this case.

Case 2. Every component $D_i(0)$, $i = 1, 2, 3, 4$, crosses every line $x = x_0$, $-\infty < x_0 < \infty$. Set

$$u_i = \text{lub } y,\ l_i = \text{glb } y \text{ for } (1, y) \in D_i(0),\ -\alpha < y < \alpha,\ i = 1, 2, 3, 4,$$

and let

$$u = \max u_1, u_2, u_3, u_4, \qquad l = \min l_1, l_2, l_3, l_4.$$

Case 2_1. $u = u_{i_1}$, $u \neq u_i$ for $i \neq i_1$. Then $l = l_{i_2}$ and by relations (3) there are at least two integers i such that $l_i \neq l$ and for at least one of them, say i_3, $i_3 \neq i_1$. Then

$$-\alpha \leqq l_{i_2} < l_{i_3} < u_{i_3} < u_{i_1} \leqq \alpha. \tag{4}$$

By (4) we have a point $(1, y_1) \in D_{i_1}(0)$, $u_{i_3} < y_1 < u_{i_1}$ and a point $(1, y_2) \in D_{i_2}(0)$, $l_{i_3} < y_2 < l_{i_3}$. Let γ^j be a simple arc with end points $(0, 0)$ and $(1, y_j)$ and lying in $D_{i_j}(0) + (0, 0)$, $j = 1, 2$. Then there exist numbers $x_1 < x_2$ such that γ^1 and γ^2 lie between the lines $x = x_1$ and $x = x_2$. If $D_{i_3}(0)$ contains a point on the line $x = x_1$ and a point on the line $x = x_2$, then a simple arc joining them and lying in $D_{i_3}(0)$ would have to cross either γ^1 or γ^2. This is impossible. Hence $D_{i_3}(0)$ satisfies the conditions of Case 1 and Case 2_1 is impossible.

Case 2_2. $l = l_{i_1}$, $l_i \neq l$ for $i \neq i_1$. The same reasoning as that used in Case 2_1 shows that Case 2_2 is impossible.

Case 2_3. $u = u_{i_1} = u_{i_2}$, $i_1 \neq i_2$ and $l = l_{i_3} = l_{i_4}$, $i_3 \neq i_4$. Then $-\alpha < l < u < \alpha$ and each component $D_i(0)$, $i = 1, 2, 3, 4$, lies in the strip $-\alpha < y < \alpha$. By the relations (3), i_1, i_2 are not both odd or both even integers and i_3, i_4 are not both odd or both even integers. By renumbering if

necessary let i_1 and i_2 be the odd integers. Then i_2 is an even integer. Take λ_0, $0<\lambda_0<\eta$. By (2), $D_{i_2}(\lambda_0)\subset D_{i_2}(0)$ and hence $D_{i_2}(\lambda_0)$ lies in the strip $-\alpha<y<\alpha$. By (2), $(1, u_{i_1})\in D_{i_1}(\lambda_0)$, $(1, l_{i_2})\in D_{i_2}(\lambda_0)$, and

$$l_{i_2} < y < u_{i_1} \qquad \text{for } (1, y) \in D_{i_2}(\lambda_0). \tag{5}$$

Let γ^1 be a simple arc with end points $(0, 0)$ and $(1, u_{i_1})$ and lying in $D_{i_1}(\lambda_0)+(0, 0)$, and let γ^2 be a simple arc with end points $(0, 0)$ and $(1, l_{i_2})$ and lying in $D_{i_2}(\lambda_0)+(0, 0)$. By (5) the same reasoning as that used in Case 2_1 shows that $D_{i_2}(\lambda_0)$ is bounded on either the right or the left.

3. **Proof of Lemma 1.2.** Let $f(x, y)$ be a real-valued function which satisfies the conditions (a), (b), (c) of §1 and assume that there is a constant $h>0$ such that $-h<f(x, y)<h$. By condition (c) there is a number $a\neq 0$ such that

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \qquad \text{for } x^2 + y^2 \leqq a^2. \tag{6}$$

For this number $a\neq 0$ set

$$\phi(x, y) = \begin{cases} (t - \sin t)/2\pi, \quad t = 2\pi(x^2 + y^2)/a^2, & \text{for } x^2 + y^2 \leqq a^2, \\ 1 & \text{for } x^2 + y^2 > a^2. \end{cases}$$

Then $\phi(x, y)$ satisfies the following conditions. (i) ϕ is of class C'' in the entire xy-plane. (ii) $\phi(0, 0)=0$. (iii) $\phi_x=\phi_y=\phi_{xx}=\phi_{xy}=\phi_{yy}=0$ for $(x, y)=(0, 0)$ and for $x^2+y^2\geqq a^2$. (iv) $0<\phi(x, y)\leqq 1$ for $(x, y)\neq(0, 0)$. Elementary computations show that in view of (6) and the conditions (i), (ii), (iii), (iv) satisfied by ϕ there is a number $\eta>0$ such that the functions

$$F(x, y; \lambda) = f(x, y) + \lambda\phi(x, y), \qquad -\eta < \lambda < \eta,$$

satisfy the following conditions.

(a′) $F(x, y; \lambda)$ is of class C'' in the entire xy-plane.

(b′) $F_{xx}F_{yy}-F_{xy}^2\leqq 0$.

(c′) $F(0, 0; \lambda)=0$, $F_x(0, 0; \lambda)=0$, $F_y(0, 0; \lambda)=f_y(0, 0)=q_0$, $F_{xx}(0, 0; \lambda)=f_{xx}(0, 0)$, $F_{xy}(0, 0; \lambda)=f_{xy}(0, 0)$, $F_{yy}(0, 0; \lambda)=f_{yy}(0, 0)$.

(d′) $|f(x, y)-F(x, y; \lambda)|<\eta$.

Then, for $\alpha=h/q_0$, the functions

$$G(x, y; \lambda) = F(x, y; \lambda) - q_0 y, \qquad -\eta < \lambda < \eta,$$

satisfy the assumptions of Lemma 2.1. Hence for the number λ_0, $0\leqq\lambda_0<\eta$, determined in Lemma 2.1, the function $f^*(x, y)=F(x, y; \lambda_0)$ satisfies the conditions of Lemma 1.2.

THE OHIO STATE UNIVERSITY

THE ALGEBRAIC CHARACTER OF A CLASS OF HARMONIC FUNCTIONS IN THREE VARIABLES[1]

P. M. PEPPER

1. Introduction. Let the harmonic function in three variables x, y, z be given by its series development into spherical harmonics, that is, let the harmonic function be given by

$$F(r, \cos\theta, \phi) = \sum_{n=0}^{\infty} \sum_{\nu=-n}^{n} A_{n\nu} r^n P_n^{(\nu)}(\cos\theta) e^{i\nu\phi}, \tag{1}$$

where spherical coordinates have been introduced by the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta\cos\phi, \qquad z = r\sin\theta\sin\phi. \tag{2}$$

It is obvious that the coefficients $A_{n\nu}$ of the series development (1) determine completely the function $F(r, \cos\theta, \phi)$. Thus the sequence of coefficients determines the analytic character of F, the locations and nature of singular points, and so on. It seems therefore natural to look for and extract those properties of a sequence of coefficients $A_{n\nu}$ which most readily yield relevant information. In the case of harmonic functions of two variables, the problem reduces immediately to that of detecting the singularities of an analytic function given by its Taylor development. This last mentioned problem, also known as the Hadamard problem, has occupied the attention of mathematicians for a considerable period and a great many results have been obtained. When dealing, however, with harmonic functions of three variables, the problem becomes much more difficult and it seems impossible to apply directly the methods developed in the theory of functions of one complex variable. We have, however, at our disposal the method of integral operators introduced by S. Bergman, which enables us to represent harmonic functions in three variables by means of an integral operator on a function of a complex variable. This representation makes it possible to carry over certain results of the well known theory of functions of one complex variable into the relatively little developed theory of harmonic functions of three variables. The present paper is to be considered as a part of an extended program of study of harmonic functions given by the de-

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velopment (1) through the above indicated means.

In the present paper we obtain sufficient conditions for (1) to represent an algebraic function, taking as a starting point the following result of S. Bergman [1].[2]

Let $g(u, \zeta)$ be a rational function in two variables u and ζ, and, putting

$$u = x + iy\left(\frac{\zeta + \zeta^{-1}}{2}\right) + z\left(\frac{\zeta - \zeta^{-1}}{2}\right), \tag{3}$$

define $H(x, y, z)$ by the integral

$$H(x, y, z) = \frac{1}{2\pi i}\int_C g(u, \zeta)d\zeta \tag{4}$$

where C is a suitable curve in the complex ζ-plane. Then $H(x, y, z)$ expressed as a function of r, $\cos\theta$, ϕ through (2) represents an algebraic harmonic function $F(r, \cos\theta, \phi)$, and, conversely, every algebraic harmonic function can be obtained from (4) with $g(u, \zeta)$ algebraic or an integral of a specified kind of an algebraic function.

Moreover, if by means of (3), $g(u, \zeta)$ is written as the quotient of two polynomials in x, y, z, and ζ without common factors

$$g(u, \zeta) = \frac{p(u, \zeta)}{q(u, \zeta)} = \frac{P(\zeta; x, y, z)}{Q(\zeta; x, y, z)}, \tag{5}$$

then the degree of $Q(\zeta; x, y, z)$ in ζ gives, in general, the multivaluedness of the algebraic function $H(x, y, z) = F(r, \cos\theta, \phi)$, that is, the number of space sheets which are necessary in order to make it uniform.

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2. Criteria for rationality of functions of two variables. In view of the foregoing it is important for us to find conditions on the coefficients $a(n, \nu)$ in the double series development of $g(u, \zeta)$,

$$g(u, \zeta) = \sum_{n=0}^{\infty}\sum_{\nu=-\infty}^{\infty} a(n, \nu)u^n\zeta^\nu,$$

which entail the rationality of $g(u, \zeta)$ in both its arguments.

Toward this end it will be found convenient to introduce the following notations. For any sequence $\{a_n\}(n=0, 1, 2, \cdots)$ we shall, using the corresponding capital letter, employ the customary nota-

[2] Numbers in brackets refer to the references cited at the end of the paper.

tion

$$A_\lambda^{(\mu)} = \begin{vmatrix} a_\lambda & a_{\lambda+1} & \cdots & a_{\lambda+\mu} \\ \cdot & & & \\ \cdot & & & \\ a_{\lambda+\mu} & a_{\lambda+\mu+1} & \cdots & a_{\lambda+2\mu} \end{vmatrix}; \quad \lambda, \mu = 0, 1, 2, \cdots$$

for the Kronecker determinants.

Furthermore, if $a(n, \nu)$ $(n, \nu = \cdots, -1, 0, 1, 2, \cdots)$ is a double sequence, we again use the corresponding capital letter in denoting the following determinants:

$$A_\lambda^{(\mu)}(k_0, k_1, \cdots, k_\mu) = \begin{vmatrix} a(\lambda, k_0) & a(\lambda+1, k_1) & \cdots & a(\lambda+\mu, k_\mu) \\ \cdot & & & \\ \cdot & & & \\ a(\lambda+\mu, k_0) & a(\lambda+\mu+1, k_1) & \cdots & a(\lambda+2\mu, k_\mu) \end{vmatrix}, \tag{6}$$

$\lambda = \cdots, -1, 0, 1, \cdots; \mu = 0, 1, 2, \cdots;$ and $k_0, k_1, \cdots, k_\mu$ being any $\mu+1$ integers.

Since our aim in this paper is merely to derive certain sufficient conditions for the coefficients of (1) in order that it represent an algebraic harmonic function, it will be enough for us to confine ourselves to functions $g(u, \zeta)$ which have the series development

$$g(u, \zeta) = \sum_{n=0}^{\infty} \sum_{\nu=-n}^{n} a(n, \nu) u^n \zeta^\nu. \tag{7}$$

Thus, whenever employing some notation such as (6) where coefficients $a(n, \nu)$ with $|\nu| > n$ appear, we invariably use the convention

$$a(n, \nu) = 0 \qquad \text{for } |\nu| > n. \tag{8}$$

The curve of integration in (4) will surround the origin and hence the series (7) has to be convergent for a sufficiently small $|u|$ and ζ on the curve. Indeed, in order to manipulate freely with the series, it is sufficient that it converge uniformly and absolutely. This curve may be taken as a circle about the origin in the ζ-plane so that the Cauchy-Hadamard criterion for uniform and absolute convergence applies. If there exists a positive and finite r such that

$$\limsup_{n \to \infty} \left(\sum_{\nu=-n}^{n} | a(n, \nu) r^\nu | \right)^{1/n} < \infty, \tag{9}$$

then the series (7) is absolutely and uniformly convergent for $|\zeta| = r$ and $|u|$ sufficiently small. There is, of course, no loss of generality

in assuming $r=1$, and in this case inequality (9) becomes lim sup $(\sum_{\nu=-n}^{n}|a(n,\nu)|)^{1/n}<\infty$. This condition is equivalent to: there exists a finite R such that

$$|a(n,\nu)|<R^n \text{ for } \nu=-n,-n+1,\cdots,n;\ n=0,1,2,\cdots. \tag{10}$$

To establish rationality of $g(u,\zeta)$ in both its arguments we establish it for u and ζ separately by means of the well known criteria first given by Kronecker [3].

If we put

$$c_n=c_n(\zeta)=\sum_{\nu=-n}^{n}a(n,\nu)\zeta^\nu \tag{11}$$

in (7), the condition for the rationality of $g(u,\zeta)$ in u becomes the Kronecker condition for the series $\sum_{n=0}^{\infty}c_nu^n$ to represent a rational function. One of the conditions given by Kronecker is that the determinants $C_0^{(\mu)}$ vanish for all sufficiently large μ.

Now, on substituting the values of c_n as given by (11) in $C_\lambda^{(\mu)}$ we obtain

$$C_\lambda^{(\mu)}=\begin{vmatrix} \sum_{\nu=-\lambda}^{\lambda}a(\lambda,\nu)\zeta^\nu & \sum_{\nu=-\lambda-1}^{\lambda+1}a(\lambda+1,\nu)\zeta^\nu & \cdots & \sum_{\nu=-\lambda-\mu}^{\lambda+\mu}a(\lambda+\mu,\nu)\zeta^\nu \\ \cdot & & & \\ \cdot & & & \\ \sum_{\nu=-\lambda-\mu}^{\lambda+\mu}a(\lambda+\mu,\nu)\zeta^\nu & \sum_{\nu=-\lambda-\mu-1}^{\lambda+\mu+1}a(\lambda+\mu+1,\nu)\zeta^\nu & \cdots & \sum_{\nu=-\lambda-2\mu}^{\lambda+2\mu}a(\lambda+2\mu,\nu)\zeta^\nu \end{vmatrix}.$$

Using the convention (8) and the notations (6) we easily see that

$$C_\lambda^{(\mu)}=\sum_{m=-(1+\mu)(\lambda+\mu)}^{(1+\mu)(\lambda+\mu)}\ \sum_{k_0+k_1+\cdots+k_\mu=m}A_\lambda^{(\mu)}(k_0,k_1,\cdots,k_\mu)\zeta^m \tag{12}$$

where the inner summation is taken over all integral values which add up to m.[3] Because of the linear independence of the powers of ζ, the requirement that $C_\lambda^{(\mu)}$ vanish identically in ζ is evidently equivalent to the vanishing of the coefficients of the different powers of ζ in (12).

If we introduce the abbreviation

$$A_\lambda^{(\mu)}\{m\}=\sum_{k_0+k_1+\cdots+k_\mu=m}^{(\mu)}A_\lambda^{(\mu)}(k_0,k_1,\cdots,k_\mu), \tag{13}$$

[3] From relation (8) it is seen that the k_ν may be restrained to satisfy $|k_\nu|\leqq\lambda+\mu+\nu$ without loss of generality, for, any $A_\lambda^{(\mu)}(k_0,k_1,\cdots,k_\mu)$ with $|k_{\nu_0}|>\lambda+\mu+\nu_0$ has a column of zeros.

this condition can be written

$$A_\lambda^{(\mu)}\{m\} = 0$$

for $m = -(\mu+1)(\lambda+\mu), -(\mu+1)(\lambda+\mu)+1, \cdots, (\mu+1)(\lambda+\mu)$.

Because of (8), all the sums appearing in (13) are finite.

We have thus established the following lemma.

LEMMA I. *A necessary and sufficient condition for $g(u, \zeta)$ to be a rational function of u is that the series development satisfies the conditions $A_0^{(\mu)}\{m\}=0$ for $m=-\mu^2-\mu,\ -\mu^2-\mu+1, \cdots, \mu^2+\mu$ and all sufficiently large μ.*

We are, however, also interested in the degrees of the numerator and denominator of $g(u, \zeta)$ and for this purpose it is better to apply another form of the Kronecker criterion. We refer to the one stating that for $\sum c_n u^n$ to represent a rational function with the denominator of degree not exceeding β and the numerator of degree not exceeding α $(\alpha \geqq \beta-1)$,[4] it is necessary and sufficient that

$$C_\lambda^{(\beta)} = 0 \qquad \text{for } \lambda = \alpha-\beta+1, \alpha-\beta+2, \cdots.$$

Using in this criterion (12) and (13) we obtain the following lemma.

LEMMA II. *A necessary and sufficient condition for $g(u, \zeta)$ to be a rational function of u with the denominator of degree not exceeding β and the numerator of degree not exceeding α is that*

$$A_\lambda^{(\beta)}\{m\} = 0 \tag{14}$$

for $m = -(\beta+1)(\lambda+\beta), -(\beta+1)(\lambda+\beta)+1, \cdots, (\beta+1)(\lambda+\beta)$ and $\lambda=\alpha-\beta+1, \alpha-\beta+2, \cdots$.

In order to apply directly the Kronecker criteria in studying the rationality of $g(u, \zeta)$ in ζ, it is convenient to introduce a new variable U through the substitution $u = U\zeta$. Simultaneous rationality in u and ζ is, of course, equivalent to simultaneous rationality in U and ζ. We then have

$$g(u, \zeta) = G(U, \zeta) = \sum_{n=0}^{\infty} \sum_{\nu=-n}^{n} a(n, \nu) U^n \zeta^{\nu+n}. \tag{15}$$

[4] The restriction $\alpha \geqq \beta-1$ in the Kronecker criterion is not essential. This may readily be shown by his methods if terms $c_n=0$ are defined for all $n<0$. Henceforth, this restriction is omitted, and $C_\lambda^{(\mu)}$ with $\lambda<0$ are employed.

Under the convergence conditions (9) and (10) imposed on the series $g(u, \zeta)$ it is possible to rearrange $G(U, \zeta)$ into

$$G(U, \zeta) = \sum_{n=0}^{\infty} \sum_{\nu=n'}^{\infty} a(\nu, n-\nu) U^{\nu} \zeta^{n}$$

where $n' = [(n+1)/2]$, the brackets denoting, as customary, the integral part.

Putting

$$b_n = b_n(U) = \sum_{\nu=n'}^{\infty} a(\nu, n-\nu) U^{\nu}; \quad n = 0, 1, 2, \cdots,$$

we have

$$G(U, \zeta) = \sum_{n=0}^{\infty} b_n \zeta^{n}$$

and the Kronecker condition for rationality becomes $B_0^{(\mu)} = 0$ for all sufficiently large μ.

We now put $a^*(n, \nu) = a(\nu, n-\nu)$ and denote by $A_\lambda^{*(\mu)}(k_0, k_1, \cdots, k_\mu)$ and $A_\lambda^{*(\mu)}\{m\}$ the quantities obtained from $A_\lambda^{(\mu)}(k_0, k_1, \cdots, k_\mu)$ and $A_\lambda^{(\mu)}\{m\}$ on replacing $a(n, \nu)$ by $a^*(n, \nu)$ in (6) and (13).

A calculation similar to the one that led to (12) gives

$$B_\lambda^{(\mu)} = \sum_{m=s'}^{\infty} A_\lambda^{*(\mu)}\{m\} U^{m}$$

where

(16) $s = s(\lambda, \mu) = (\lambda + \mu)(\mu + 1)$ and $s' = s'(\lambda, \mu) = [(s+1)/2]$.

It should be remarked that if $k_0 + k_1 + \cdots + k_\mu < s'$, then $A_\lambda^{*}(\mu)(k_0, k_1, \cdots, k_\mu) = 0$ as a consequence of (8), for in that case each of the products in the complete expansion of the determinant has at least one zero factor.[5]

[5] Suppose that $a^*(\lambda+\nu_0, k_0)a^*(\lambda+1+\nu_1, k_1) \cdots a^*(\lambda+\mu+\nu_\mu, k_\mu)$ is a nonvanishing term in the complete expansion, then $\nu_0, \nu_1, \cdots, \nu_\mu$ is a permutation of $0, 1, \cdots, \mu$. From (8) and $a^*(n, \nu) = a(\nu, n-\nu)$ it follows that $a^*(n, \nu) \neq 0$ only if $\nu \geqq n/2$. When this is applied to the determinant term, we see that $k_0 \geqq (\lambda+\nu_0)/2$, $k_1 \geqq (\lambda+1+\nu_1)/2$, $\cdots$, $k_\mu \geqq (\lambda+\mu+\nu_\mu)/2$. Addition of these inequalities yields $k_0+k_1+\cdots+k_\mu \geqq (\lambda+\mu)(\mu+1)/2$. The left member of this inequality is an integer and the right member is s' except when λ is odd and μ is even, in which case it is $s'-1/2$. Since s' is an integer, also, this implies that $k_0+k_1+\cdots+k_\mu$ is at least as great as s' even in this case. This permits s' to be introduced as a lower bound of the integers m for which the test of Lemma III (below) must be applied, all smaller values being irrelevant.

Thus, taking into account the convergence conditions (9) and (10) we may formulate the following lemma.

LEMMA III. *A necessary and sufficient condition for* $G(U, \zeta)$ *to be a rational function in* ζ *is that the coefficients* $a(n, \nu)$ *satisfy*

$$A_0^{*(\mu)}\{m\} = 0 \quad \text{for integral } m \geqq (\mu^2+\mu)' = [(\mu^2+\mu+1)/2] \tag{17}$$

and all sufficiently large μ.

Again, when trying to obtain the degrees of the numerator and denominator it is useful to employ the other Kronecker criterion and thus obtain

LEMMA IV. *A necessary and sufficient condition for* $G(U, \zeta)$ *to be rational in* ζ *with degree of denominator not exceeding* δ *and degree of numerator not exceeding* γ (*cf. footnote* 2) *is that*

$$A_\lambda^{*(1)}\{m\} = 0 \tag{18}$$

for integral $m \geqq s'(\lambda, \delta)$; $\lambda = \gamma-\delta+1, \gamma-\delta+2, \cdots$, *where the function* s' *is given by* (16). *In particular, the truth of* (18) *for sufficiently large* λ *is a necessary and sufficient condition on* $G(U, \zeta)$ *that its denominator be of degree not greater than* δ *in* ζ.

3. **Application to harmonic functions.** On taking the path of integration in (4) as, say, the circle $|\zeta| = 1$, the term $a(n, \nu)u^n\zeta^\nu$ will for $|\nu| \leqq n$ yield $A_{n,\nu}r^nP_n^{(\nu)}(\cos\theta)e^{i\nu\phi}$ with $A_{n,\nu} = a(n, \nu)\ C_{2n,n+\nu}i^\nu/2^n$, where $C_{2n,n+\nu}$ denotes the binomial coefficient. From the fact that the integral operators operating on a rational function yield algebraic harmonic functions, we obtain the following theorem.

THEOREM I. *Let the series* (1) *of spherical harmonics satisfy for some finite* R

$$|A_{n,\nu}| < R^n \quad \text{for } \nu = -n, -n+1, \cdots, n;\ n = 1, 2, 3, \cdots, \tag{19}$$

and, on putting

$$a(n, \nu) = \frac{2^n(n+\nu)!(n-\nu)!}{(2n)!i^\nu}A_{n,\nu} \qquad \text{for } |\nu| \leqq n, \tag{20}$$

$$a(n, \nu) = 0 \qquad \text{otherwise,}$$

let the following conditions be satisfied:

$$A_0^{(\mu)}\{m\} = 0 \qquad \text{for } m = -\mu^2-\mu, -\mu^2-\mu+1, \cdots, \mu^2+\mu$$

and

$$A_0^{*(\mu)}\{m\} = 0 \qquad \text{for integral } m \geqq [(\mu^2+\mu+1)/2]$$

for all μ sufficiently large. Then $F(r, \cos\theta, \phi)$ is an algebraic function.

The condition (19) is easily seen by (20) to be the counterpart of assumption (10) which allowed us to rearrange the series, integrate term by term, and so on.

Using Lemmas II and IV, and the last results of Bergman mentioned in the introduction we can obtain some more detailed information. If α and β are the degrees of numerator and denominator of $G(U, \zeta)$ in U and γ and δ the respective degrees in ζ, then, since $U\zeta^2$ is a polynomial in x, y, z, and ζ of degree 2 in ζ, it follows that the degree in ζ of $Q(\zeta; x, y, z)$ of (5) is at most equal to $2 \max(\alpha, \beta)+\delta$. (Cf. (15).) Thus we obtain the following theorem.

THEOREM II. *Let the series of spherical harmonics* (1) *satisfy the condition* (19) *and furthermore, let there exist non-negative integers α, β, δ such that the $a(n, \nu)$ defined by* (20) *satisfy*

$$A_\lambda^{(\beta)}\{m\} = 0$$

for $m = -(\beta+1)(\lambda+\beta), -(\beta+1)(\lambda+\beta)+1, \cdots, (\beta+1)(\lambda+\beta)$; $\lambda = \alpha-\beta+1, \alpha-\beta+2, \cdots$, *and*

$$A_\lambda^{*(\delta)}\{m\} = 0$$

for integral $m \geqq s'(\lambda, \delta)$ (cf. (16)) *for all sufficiently large λ, then $F(r, \cos\theta, \phi)$ is an algebraic function which is at most* $(2 \max(\alpha, \beta)+\delta)$*-valued.*

4. Remarks. The results of this paper may be extended and developed in several directions.

The coefficients $A_{n\nu}$ of (1) are obtained directly from those coefficients of $\sum a(n, \nu)u^n\zeta^\nu$ for which $|\nu| \leqq n$, and that is why we limited ourselves to the consideration of series $g(u, \zeta)$ which satisfy $a(n, \nu) = 0$ for $|\nu| > |n|$. However, the conditions of rationality of power series depend on all their coefficients and thus it would be interesting to find out not only when (7) represents a rational function but also when it can be extended, by introducing terms with $|\nu| > n$ into doubly infinite series representing rational functions. This would give less strict sufficient conditions for the algebraic character of (1).

These conditions would, however, be still very far from necessary and it would be interesting to derive from Bergman's result men-

tioned in the introduction some necessary conditions for the algebraic character of (1).

Another direction in which it is possible to extend our results is to try to carry over through the integral operators into the theory of harmonic functions more detailed results of the theory of power series, that is, those of Hadamard, Mandelbrojt, Dvoretzky, and others relating to the location and nature of the singularities of analytic functions of one complex variable, into the theory of harmonic functions of three variables.

Finally, the method of integral operators has been extended by Bergman [3] to much more general partial differential equations than the Laplacian and thus it is possible to extend our results on harmonic functions to the solutions of these general partial differential equations.

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Harvard University and
University of Notre Dame

CERTAIN INVERSION FORMULAS FOR THE LAPLACE TRANSFORM

DENNIS B. AMES

1. Introduction. In this paper certain inversion formulas are obtained for the Laplace transform. The case of a periodic determining function is first treated and some conclusions are reached about such functions when their transforms vanish at points in an arithmetical progression on a vertical line. This yields a contrast with Lerch's theorem. Later an inversion formula is derived for the more general case when the determining function is not assumed to be periodic but does have a finite abscissa of absolute convergence.

Let $F(q)$ be a real function of the real variable q whose Laplace transform, $f(x+iy)=u(x, y)+iv(x, y)$, is convergent for $x>c$. Then

$$(1)\qquad u(x, y) = \int_0^\infty F(q)\cos yq e^{-xq}dq, \qquad v(x, y) = -\int_0^\infty F(q)\sin yq e^{-xq}dq.$$

If the transform of $F(q)$ is known, then (1) give the real transforms of $F(q)\cos yq$ and $F(q)\sin yq$. Define $\phi(x, t)=\int_0^t F(q)e^{-xq}dq$ for $t>0$ and $\phi(x, 0)=0$. Then $\phi(x, t)$ is a continuous function of t for $t\geqq 0$.

2. Heuristic treatment of the periodic case. Let $F(q)$ be periodic with the period $2k$. Then for $x>0$

$$(2)\qquad (1-e^{-2kx})f(x) = \int_0^{2k} F(q)e^{-xq}dq,$$

$$(1-e^{-2kx})u\left(x, \frac{n\pi}{k}\right) = \int_0^{2k} F(q)\cos\frac{n\pi q}{k}e^{-xq}dq,$$

$$(1-e^{-2kx})v\left(x, \frac{n\pi}{k}\right) = -\int_0^{2k} F(q)\sin\frac{n\pi q}{k}e^{-xq}dq.$$

Let $a_0/2+\sum_{n=1}^\infty (a_n\cos(n\pi t/k)+b_n\sin(n\pi t/k))$ be the Fourier series of the function of period $2k$ which is equal to $\phi(x, t)$ for $0<t<2k$. If inversion of the order of integration is permissible then we have

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$$\frac{a_0}{2} = \frac{1}{2k}\int_0^{2k} dt \int_0^t F(q)e^{-zq}dq$$
$$= \frac{1}{2k}\int_0^{2k} F(q)e^{-zq}dq \int_q^{2k} dt$$
$$= f(x) + \frac{1}{2k}(1 - e^{-2kz})f'(x).$$
$$a_n = \frac{1}{k}\int_0^{2k} \cos\frac{n\pi t}{k}\,dt \int_0^t F(q)e^{-zq}dq$$
$$= \frac{1}{k}\int_0^{2k} F(q)e^{-zq}dq \int_q^{2k} \cos\frac{n\pi t}{k}\,dt$$
$$= \frac{1}{n\pi}(1 - e^{-2kz})v\left(x, \frac{n\pi}{k}\right),$$

and similarly

$$b_n = \frac{1}{n\pi}\left[1 - e^{-2kz}\right]\left[u\left(x, \frac{n\pi}{k}\right) - f(x)\right].$$

Thus if the function $\phi(x, t)$ satisfies any set of conditions which ensure the convergence of its Fourier series and its representation by this series, then we would formally obtain for $x>0$ and $0<t<2k$ the following inversion formula

$$\int_0^t F(q)e^{-zq}dq = f(x) + \frac{1}{2k}(1 - e^{-2kz})f'(x)$$
$$+ \frac{1}{\pi}(1 - e^{-2kz})\sum_{n=1}^{\infty}\frac{1}{n}\left\{v\left(x, \frac{n\pi}{k}\right)\cos\frac{n\pi t}{k}\right.$$
$$\left. + \left[u\left(x, \frac{n\pi}{k}\right) - f(x)\right]\sin\frac{n\pi t}{k}\right\}. \tag{3}$$

3. **Proof of the inversion formula for the periodic case.** The foregoing heuristic argument suggests then a study of the expression

$$K = \frac{1}{\pi}(1 - e^{-2kz})\sum_{n=1}^{\infty}\frac{1}{n}\left\{v\left(x, \frac{n\pi}{k}\right)\cos\frac{n\pi t}{k}\right.$$
$$\left. + \left[u\left(x, \frac{n\pi}{k}\right) - f(x)\right]\sin\frac{n\pi t}{k}\right\}$$

for $x>0$ and $0<t<2k$.

Call $H=f(x)+(1/2k)\ (1-e^{-2kx})f'(x)$. By use of (2) it is seen that

$$K=\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2k}F(q)e^{-xq}dq\,\frac{1}{n}\left[\sin\frac{n\pi(t-q)}{k}-\sin\frac{n\pi t}{k}\right]$$

with $x>0$ and $0<t<2k$.

Now the series $\sum_{n=1}^{\infty}(1/n\pi)\sin(n\pi\theta/k)$ is uniformly convergent to $(1/2k)(k-\theta)$ for $0<\theta<2k$ and is boundedly convergent in $0\leqq\theta\leqq 2k$. In fact the partial sums are less than or equal to $1/\pi+1/2$ for all n and all θ. Hence inversion of summation and integration is valid. This is true when the factor $F(q)e^{-xq}$ is integrable but not necessarily bounded. If we write K as

$$K=\int_0^t F(q)e^{-xq}dq\sum_{n=1}^{\infty}\frac{1}{n\pi}\left[\sin\frac{n\pi(t-q)}{k}-\sin\frac{n\pi t}{k}\right]$$
$$-\int_t^{2k}F(q)e^{-xq}dq\sum_{n=1}^{\infty}\frac{1}{n\pi}\left[\sin\frac{n\pi(q-t)}{k}+\sin\frac{n\pi t}{k}\right],$$

it is seen that the first sum on the right is $q/2k$, $0<q<t<2k$, while the second sum is $(1/2k)\ (2k-q)$, $0<t<q<2k$. Hence

$$K=\frac{1}{2k}\int_0^{2k}qF(q)e^{-xq}dq-\int_t^{2k}F(q)e^{-xq}dq$$
$$=\frac{-1}{2k}\frac{d}{dx}\left[(1-e^{-2kx})f(x)\right]-\int_t^{2k}F(q)e^{-xq}dq.$$

Thus $H+K=\int_0^t F(q)e^{-xq}dq$. Hence we have proved:

THEOREM I. *Let $F(q)$ have a Laplace transform given by $f(x+iy)=u(x,y)+iv(x,y)$. Let $F(q)$ have the period $2k$. Then for $x>0$ and $0<t<2k$, the inversion formula* (3) *is true.*

The formula (3) can be written

$$\int_0^t F(q)e^{-xq}dq$$
$$=f(x)+\left\{1-e^{-2kx}\right\}\left\{\frac{1}{2k}f'(x)-\frac{1}{2k}(k-t)f(x)\right.$$
$$\left.+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[v\left(x,\frac{n\pi}{k}\right)\cos\frac{n\pi t}{k}+u\left(x,\frac{n\pi}{k}\right)\sin\frac{n\pi t}{k}\right]\right\} \tag{3a}$$

for $0<t<2k$.

4. A second inversion formula. Consider the expression

$$W = \frac{1}{k}\{1 - e^{-2kx}\}\left\{\frac{f(x)}{2} + \sum_{n=1}^{\infty}\left[u\left(x, \frac{n\pi}{k}\right)\cos\frac{n\pi t}{k} - v\left(x, \frac{n\pi}{k}\right)\sin\frac{n\pi t}{k}\right]\right\}.$$

This expression is obtained formally by differentiating the right side of (3) with respect to t. Let $F(q)$ be of period $2k$. Then by use of (2) it can be seen that W is the Fourier series of the function given by $G(t) = F(t)e^{-xt}$, $0<t<2k$, $G(t+2k) = G(t)$. We thus have:

THEOREM II. *Let $F(q)$ have a Laplace transform given by $f(x+iy) = u(x, y)+iv(x, y)$. Let it have the period $2k$. If the function $F(t)e^{-xt}$, $x>0$, $0<t<2k$, satisfies any set of conditions which ensure the convergence of its Fourier series to the function, then the following inversion formula is true:*

$$F(t) = \frac{1}{k}e^{xt}[1 - e^{-2kx}]\left[\frac{1}{2}f(x) + \sum_{n=1}^{\infty}\left(u\left(x, \frac{n\pi}{k}\right)\cos\frac{n\pi t}{k} - v\left(x, \frac{n\pi}{k}\right)\sin\frac{n\pi t}{k}\right)\right] \tag{4}$$

for $0<t<2k$. If $F(t)$ is of bounded variation in the neighborhood of the point t, then the $F(t)$ on the left of the formula represents $(F(t+0) + F(t-0))/2$.

If $F(t)$ is complex, $F(t) = F_1(t)+iF_2(t)$, then it is easy to see that an inversion formula similar to (4) exists, since (4) exists for each of F_1 and F_2. For F_1 and F_2 have individually the period $2k$ if $2k$ is real. If $F(t)$ is a nonperiodic function which is zero for $t>t_0$, then an inversion formula similar to (4) exists. The modifications are to suppress the factor $1-e^{-2kx}$ and replace $2k$ by t_0.

5. **Some properties of the periodic case.** Let $F(t)$ be real and have the real period $2k$. Let its transform for some $x>0$ vanish at the points $x+in\pi/k$ ($n=1, 2, 3, \cdots$) of a vertical line. Then $u(x, n\pi/k)$ and $v(x, n\pi/k)$ are zero and it is seen by (3a) that

$$F(t) = \frac{1}{2k}(1 - e^{-2kx})f(x)e^{xt}, \qquad 0 < t < 2k. \tag{5}$$

For a given transform $f(x+iy)$ this is the unique real periodic function. Of course a periodic function of period $2k$ of the form $G(x)e^{xt}$, $0<t<2k$, where G is arbitrary would have a transform which vanishes at the same set of points. Since $u(x, y)$ and $v(x, y)$ are respectively

even and odd functions of y the transform would also vanish at the points where n is a negative integer of the same vertical line.

If in addition the transform vanishes at the point x on the real axis then it is seen that $F(t)=0$. Hence:

THEOREM. *A necessary and sufficient condition that the periodic function $F(t)$ of period $2k$ be null is that its transform vanish at the points $x+in\pi/k$ $(n=0, 1, 2, \cdots)$ of a vertical line.*

This theorem is true when F is complex, as can be seen at once when the inversion formula for this case is set up as described in the previous section. Moreover if the transform vanishes only at the points $x+in\pi/k$ $(n=1, 2, 3, \cdots)$, then $F(t)$ is again given by (5).

More generally let $F(t)=F_1+iF_2$ and have the real period $2k$. Let its transform f vanish at the points $x+iy+in\pi/k$ $(n=0, \pm 1, \pm 2, \cdots)$ of a vertical line where y is not necessarily zero. Separating $f(x+iy+in\pi/k)=0$ into real and imaginary parts we have

$$\int_0^{2k} e^{-xq}\left[F_1\cos\left(y+\frac{n\pi}{k}\right)q+F_2\sin\left(y+\frac{n\pi}{k}\right)q\right]dq=0,$$

$$\int_0^{2k} e^{-xq}\left[F_2\cos\left(y+\frac{n\pi}{k}\right)q-F_1\sin\left(y+\frac{n\pi}{k}\right)q\right]dq=0.$$

Writing $-n$ for n in each of these formulas and adding and subtracting the results for each formula, we obtain

$$\int_0^{2k} e^{-xq}[F_2\sin yq+F_1\cos yq]\begin{matrix}\sin\\ \cos\end{matrix}\frac{n\pi q}{k}\,dq=0,$$

$$\int_0^{2k} e^{-xq}[F_2\cos yq-F_1\sin yq]\begin{matrix}\sin\\ \cos\end{matrix}\frac{n\pi q}{k}\,dq=0,$$

$n=0, \pm 1, \pm 2, \cdots$. That $n=0$ is included is seen from the previous pair of equations. Hence if F is bounded and has at most a finite number of discontinuities in the interval $0<q<2k$, we obtain at each point of continuity

$$F_2\sin yq+F_1\cos yq=0,\ F_2\cos yq-F_1\sin yq=0.$$

The determinant equals -1, hence $F_1=0$ and $F_2=0$ at each point of continuity. Hence F is a null function. We thus have the following theorem true whether F is complex or real.

THEOREM. *A necessary and sufficient condition that a periodic function of period $2k$ be null is that its transform vanish at the points $x+iy+in\pi/k$ $(n=0, \pm 1, \pm 2, \cdots)$ of a vertical line.*

It is seen that the theorem is true if the common distance of the equidistant points on the vertical line has the form π/mk where m is an integer. The following example shows that if this distance does not have this form, then $F(t)$ is not necessarily null. Let $F=-1$, $0<t<k$; $F=e^k$, $k<t<2k$; $F(t+2k)=F(t)$. The transform of this function is $(e^k-e^{ks})/s(1+e^{ks})$. It vanishes at the points $s=1+2n\pi i/k$ $(n=0, \pm 1, \pm 2, \cdots)$. Here the common distance on the vertical line $x=1$ is $2\pi/k$.

6. **Proof of an inversion formula for the general case.** Let $F(q)$ have a Laplace transform which is absolutely convergent for $x>c$. It is of interest to see that the foregoing results lead to an inversion formula for this case where F is not assumed periodic. The formula has a form similar to that of the periodic case. The fact that $H+K$ in §3 is independent of k leads us quite naturally to a consideration of the expression W, defined below, as $k\to\infty$. For a fixed positive t and for $x>c$ set up the following expression in which k has any value for which $0<t<2k$,

$$W=\sum_{n=1}^{\infty}\frac{1}{n\pi}\left[v\left(x,\frac{n\pi}{k}\right)\cos\frac{n\pi t}{k}+u\left(x,\frac{n\pi}{k}\right)\sin\frac{n\pi t}{k}\right]$$

$$=\sum_{n=1}^{\infty}\frac{1}{n\pi}\int_0^{\infty}F(q)e^{-xq}\sin\frac{n\pi(t-q)}{k}\,dq.$$

Decompose the integral into three integrals: W_1 from 0 to t, W_2 from t to $2k$, and W_3 from $2k$ to ∞, so that

$$W=\sum_{n=1}^{\infty}\frac{1}{n\pi}(W_1+W_2+W_3).$$

From the discussion in §3 we see that

$$\sum_{n=1}^{\infty}\frac{1}{n\pi}W_1=\frac{1}{2k}\int_0^{t}F(q)e^{-xq}(k-t+q)dq,$$

$$\sum_{n=1}^{\infty}\frac{1}{n\pi}W_2=\frac{-1}{2k}\int_t^{2k}F(q)e^{-xq}(k-q+t)dq.$$

Hence as $k\to\infty$ the limits of these are

$$\frac{1}{2}\int_0^{t}F(q)e^{-xq}dq \quad\text{and}\quad \frac{-1}{2}\int_t^{\infty}F(q)e^{-xq}dq$$

respectively. We now show that $\lim_{k\to\infty}\sum_{n=1}^{\infty}1/n\pi W_3=0$.

$$\sum_{n=1}^{\infty} \frac{1}{n\pi} W_3 = -\sum_{n=1}^{\infty} \frac{1}{n\pi} \int_{2k}^{\infty} F(q)e^{-xq} \sin \frac{n\pi(q-t)}{k}\, dq.$$

Decompose the interval $(2k, \infty)$ into the subintervals $(2k, 2k+t)$, $(2k+t, 4k+t), \cdots, (2mk+t, 2mk+2k+t), \cdots$ where $m=1, 2, 3, \cdots$. In the subinterval $(2k, 2k+t)$ we have $0<2k-t<q-t<2k$, while in $(2mk+t, 2mk+2k+t)$ we have $2mk<q-t<2mk+2k$. Now

$$\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi\theta}{k} = \frac{1}{2k}\left[(2m+1)k - \theta\right], \qquad 2mk < \theta < 2(m+1)k.$$

In each subinterval the series involved is boundedly convergent and uniformly convergent in any closed interval within the integration limits. Hence the operations of summation and integration are commutable. It is seen then that

$$\sum_{n=1}^{\infty} \frac{1}{n\pi} W_3 = \frac{1}{2k}\int_{2k}^{\infty} (t-q)F(q)e^{-xq}dq + \frac{1}{2}\int_{2k}^{2k+t} F(q)e^{-xq}dq + \frac{1}{2}\sum_{m=1}^{\infty}(2m+1)\int_{2mk+t}^{2(m+1)k+t} F(q)e^{-xq}dq.$$

The first two terms on the right have limits of zero as $k\to\infty$. Consider the last term. Now for q sufficiently large, $|F(q)| < Ae^{x_0 q}$ for $x_0 > c$. For a given $x > c$ choose an x_0 such that $x > x_0 > c$. We have then for sufficiently large k, calling $x - x_0 = p > 0$, and designating this last term by T,

$$|T| < \frac{Ae^{-pt}}{2p}(1-e^{-2kp})\sum_{m=1}^{\infty}(2m+1)e^{-2mkp} = \frac{Ae^{-pt}(3-e^{-2kp})}{2p(e^{2kp}-1)}.$$

Hence T has a limit of zero as $k\to\infty$. Combining our results we have

$$\lim_{k\to\infty} W = \frac{1}{2}\int_0^t F(q)e^{-xq}dq - \frac{1}{2}\int_t^{\infty} F(q)e^{-xq}dq = \int_0^t F(q)e^{-xq}dq - \frac{1}{2}f(x).$$

We have proved:

THEOREM. *Let $F(q)$ be real and let its Laplace transform $f(x+iy)$*

$=u(x, y)+iv(x, y)$ be absolutely convergent for $x>c$. Then for $x>c$ and $t>0$

$$\int_0^t F(q)e^{-xq}dq = \frac{1}{2}f(x) + \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{1}{n\pi}\left[u\left(x, \frac{n\pi}{k}\right)\sin\frac{n\pi t}{k} + v\left(x, \frac{n\pi}{k}\right)\cos\frac{n\pi t}{k}\right]. \tag{6}$$

As an example of the formula consider $f(s)=1/(s+1)$ and take $x=0$. We obtain

$$\int_0^t F(q)dq = \frac{1}{2} + \lim_{k\to\infty}\sum_{n=1}^{\infty}\left[\frac{k^2\sin n\pi t/k}{n\pi(k^2+n^2\pi^2)} - \frac{k\cos n\pi t/k}{(k^2+n^2\pi^2)}\right].$$

For any t, provided k is large enough, we have for the sum of the sine series $[(1-t/k)-\sinh(k-t)/\sinh k]/2$. Its limit is $(1-e^{-t})/2$. The sum of the cosine series is $[\cosh(k-t)/\sinh k-1/k]/2$. Its limit is $e^{-t}/2$. Hence $\int_0^t F(q)dq=1-e^{-t}$ and $F(t)=e^{-t}$. In the proof of the above theorem the following hypotheses suffice for its truth:

(i) F is real and integrable, $t\geqq 0$. (It is not assumed bounded.)

(ii) $\int_0^\infty Fe^{-st}dt$ is absolutely convergent, Real $s=x>c$.

Hence with no implication that the transform is analytic it follows from (6) that if the transform vanishes on a vertical line in the half-plane Real $s>c$ then F is null and the transform is identically zero.

RENSSELAER POLYTECHNIC INSTITUTE

SOME PROPERTIES OF ANNULAR BLANKETS

C. A. HAYES, JR. AND A. P. MORSE

1. Introduction. Given a metric space $\mathfrak{S}$ and a set $A \subset \mathfrak{S}$, suppose that there is a function F which correlates to each point x in A such a family of nonvacuous sets that there are members of the family of diameter less than any preassigned positive number, shrinking down upon, but not necessarily containing, the point x. Such a function is called a *blanket*[1] with domain A. It is known that if a blanket possesses suitable covering properties, then almost everywhere differentiation is possible with respect to it.[2]

We deal here with a special type of blanket, which we term an *annular* blanket. We are interested in learning what effect the ability to differentiate with respect to these annular blankets has upon their covering properties. In this respect, we supplement two earlier papers by A. P. Morse.[3] In §4, we give necessary and sufficient conditions that an annular blanket in a separable space be *strong* with respect to a given measure. In §5, we restrict ourselves to the euclidean plane and utilize the properties of a special plane set[4] $\mathcal{A}$ together with the results of §4, to construct a blanket[5] whose member sets are closed, nearly circular, and *star-shaped* about a central point, with respect to which differentiation is impossible for a certain measure Ω. The measure Ω is obtained by constructing a certain plane set B_0 and defining $\Omega(\beta) = L(B_0\beta)$, where L denotes Carathéodory linear measure in the plane.

A theorem of A. S. Besicovitch[6] insures that any blanket, which correlates with a point the closed circular disks concentric about it, may be employed for purposes of differentiation for a wide class of measures, which includes Ω above. This result has been shown in

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[1] Unfamiliar technical terms are defined in §2 and §5.

[2] See A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235. Hereinafter this will be referred to as M1.

[3] See M1, and in addition, A. P. Morse, *Perfect blankets*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 418–442. Hereinafter this will be referred to as M2.

[4] See A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305. The set $\mathcal{A}$ is defined and some of its properties are given in pp. 285–286 of that paper. Hereinafter, that paper will be referred to as RM.

[5] Reference to such a blanket is made in pp. 431–432 of M2.

[6] A. S. Besicovitch, *A general form of the covering principle and relative differentiation of additive functions*, Proc. Cambridge Philos. Soc. vol. 41 (1945) pp. 103–110.

M2 to hold for a class of blankets known as *star* blankets, which includes the circular blankets just mentioned, and also certain types of nearly circular star-shaped blankets, among others. However, our results prove that not all kinds of nearly circular star-shaped blankets are suitable for differentiation.

2. **Preliminaries.** We now set forth certain notations, conventions, definitions, and other pertinent information which we shall have occasion to use.

Let us agree that $(x \in A) = (x \text{ is a member of } A) = (x \text{ is in } A)$; $(x \notin A) = (x \text{ is not a member of } A) = (x \text{ is not in } A)$; $(A \subset B) = (A \text{ is a subset of } B) = (B \supset A) = (B \text{ contains } A)$; the integer 0 and the empty set are identical.

If a_x is a set for each $x \in B$ then

$$\sum_{x \in B} a_x = \underset{y}{E} [y \in a_x \text{ for some } x \in B].$$

If $\mathfrak{F}$ is a family, then

$$\sigma\mathfrak{F} = \sum_{\beta \in \mathfrak{F}} \beta.$$

Thus $x \in \sigma\mathfrak{F}$ if and only if x is a member of some set belonging to $\mathfrak{F}$.

If $a_x \geqq 0$ for x in a countable set B, then

$$\sum_{x \in B} a_x$$

is also the appropriate numerical sum. Whether numerical summation or set theoretic summation is intended in a particular instance will be clear from the context.

2.1. DEFINITION.

$$\operatorname{sng} x = \underset{y}{E} [y = x].$$

Thus sng x is the set whose sole member is x.

We fix throughout this paper a separable metric space $\mathcal{S}$ metrized with ρ. Hereafter such words as closed, open, Borel, and so on, will refer to the metric ρ.

2.2. DEFINITION.

$$\mathfrak{C}(x, r) = \underset{y}{E} [\rho(x, y) \leqq r].$$

$\mathfrak{C}(x, r)$ is the closed sphere with center x and radius r whenever $x \in \mathcal{S}$ and $r \geqq 0$.

2.3. DEFINITION. For $A \subset \mathcal{S}$ we define

$$\text{diam } A = \sup_{(x\in A)(y\in A)} \rho(x, y) \quad \text{whenever } A \neq 0;$$

$$\text{diam } A = 0 \quad \text{whenever } A = 0.$$

2.4. DEFINITION. For $A \subset \mathfrak{S}$ we define

$$\text{nlrg } A = \underset{y}{E} [\rho(x, y) \leqq 2 \text{ diam } A \text{ for some } x \text{ in } A].$$

2.5. DEFINITION. ϕ *measures* $\mathfrak{S}$ if and only if ϕ is such a function on $E_\beta[\beta \subset \mathfrak{S}]$ to E_t $[0 \leqq t \leqq \infty]$ that

$$\phi(A) \leqq \sum_{\beta\in\mathfrak{F}} \phi(\beta)$$

whenever $\mathfrak{F}$ is a countable family for which $A \subset \sigma\mathfrak{F} \subset \mathfrak{S}$. Bearing in mind that empty sums are zero, it follows that if ϕ measures $\mathfrak{S}$ then:

I. $\phi(0) = 0$;

II. $\phi(B) \leqq \phi(A)$ whenever $B \subset A \subset \mathfrak{S}$;

III. $\phi(\sigma\mathfrak{F}) \leqq \sum_{\beta\in\mathfrak{F}} \phi(\beta)$ whenever $\mathfrak{F}$ is a countable family for which $\sigma\mathfrak{F} \subset \mathfrak{S}$.

2.6. DEFINITION. The family $\mathfrak{U}$ is defined by: $\phi \in \mathfrak{U}$ if and only if ϕ so measures $\mathfrak{S}$ that

$$\phi(A + B) = \phi(A) + \phi(B)$$

whenever $\inf_{(x\in A)(y\in B)} \rho(x, y) > 0$.

2.7. DEFINITION.

$$\mathcal{U} = \mathfrak{U} \cdot \underset{\phi}{E} [\phi(\beta) < \infty \text{ for each bounded set } \beta \subset \mathfrak{S}].$$

2.8. DEFINITION. A is *ϕ-measurable* if and only if ϕ so measures $\mathfrak{S}$ that

$$\phi(T) = \phi(TA) + \phi(T - A)$$

for each subset T of $\mathfrak{S}$.

2.9. DEFINITION. The limit notations

$$\underset{\mathfrak{F}\ni\beta\to x}{\text{limalinf}} f(\beta), \qquad \underset{\mathfrak{F}\ni\beta\to x}{\text{limalsup}} f(\beta)$$

are defined, respectively, as

$$\lim_{t\to 0+} \inf_{\beta\in H(t)} f(\beta), \qquad \lim_{t\to 0+} \sup_{\beta\in H(t)} f(\beta),$$

where

$$H(t) = \mathfrak{F} \cdot \underset{\beta}{E} [\text{diam } (\beta + \text{sng } x) < t].$$

We agree that

$$\operatorname*{limal}_{\mathfrak{F}\exists\beta\to x} f(\beta) = \operatorname*{limalsup}_{\mathfrak{F}\exists\beta\to x} f(\beta)$$

if and only if

$$\operatorname*{limalsup}_{\mathfrak{F}\exists\beta\to x} f(\beta) = \operatorname*{limalinf}_{\mathfrak{F}\exists\beta\to x} f(\beta).$$

2.10. DEFINITION. We say F is a *blanket* if and only if F is such a function that for x in its domain

(i) $x\in\mathcal{S}$ and $F(x)$ is a family of non-vacuous subsets of $\mathcal{S}$;

(ii) $\beta\in F(x)$ implies diam $\beta<\infty$;

(iii) $\inf_{\beta\in F(x)}$ diam $(\beta+\text{sng } x)=0$.

2.11. DEFINITION. If F is a blanket with domain A then the family

$$\sum_{x\in A} F(x)$$

is called the *spread* of F.

2.12. DEFINITION. F is a *natural* blanket if and only if F is a blanket and $x\in\beta$ whenever $\beta\in F(x)$.

2.13. DEFINITION. F is a *ϕ-diametrically regular* blanket if and only if $\phi\in\mathcal{U}$ and F is such a natural blanket that

$$\operatorname*{limalsup}_{F(x)\exists\beta\to x}\frac{\phi(\text{nlrg }\beta)}{\phi(\beta)}<\infty$$

for ϕ almost all x in the domain of F.

2.14. DEFINITION. F is a *Borelian* blanket if and only if F is such a blanket that β is a Borel set whenever $\beta\in F(x)$.

2.15. DEFINITION. F is a *close* blanket if and only if F is such a blanket that β is a closed set whenever $\beta\in F(x)$.

2.16. DEFINITION. F is a *ϕ-heavy* blanket if and only if F is a Borelian blanket, $\phi\in\mathcal{U}$, and the spread of F contains a countable disjointed family which covers ϕ almost all of the domain of F.

2.17. DEFINITION. G is a *subblanket* of F if and only if F and G are such blankets that $G(x)\subset F(x)$ whenever x is in the domain of G.

2.18. DEFINITION. F is a *ϕ-strong* blanket if and only if F is such a blanket that each of its subblankets is ϕ-heavy.

2.19. DEFINITION. F is a *perfect* blanket if and only if F is ϕ-strong for each $\phi\in\mathcal{U}$.

2.20. DEFINITION.

$$D(f,\phi,F,x)=\operatorname*{limal}_{F(x)\exists\beta\to x}\frac{f(\beta)}{\phi(\beta)}.$$

The next theorem is a consequence of 6.11 of M1 and the one immediately following it is quoted directly from 8.11 of M1.

2.21. THEOREM. *Every ϕ-diametrically regular close blanket is ϕ-strong.*

2.22. THEOREM. *If F is ϕ-strong and $f \in \mathcal{U}$, then $0 \leqq D(f, \phi, F, x) < \infty$ for ϕ almost all x in the domain of F.*

2.23. DEFINITION. F is a ***universal*** blanket if and only if F is such a blanket that whenever $f \in \mathcal{U}$ and $\phi \in \mathcal{U}$, then $0 \leqq D(f, \phi, F, x) < \infty$ for ϕ almost all x in the domain of F.

A perfect blanket is 2.22 evidently a universal blanket.

2.24. DEFINITION. F is *ϕ-free* if and only if $\phi \in \mathcal{U}$ and F is such a blanket with domain A that for each Borel set $B \subset \mathfrak{S}$,

$$\operatorname*{limal}_{F(x) \exists \beta \to 0} \frac{\phi(B\beta)}{\phi(\beta)} = 0$$

for ϕ almost all $x \in A \cdot (\mathfrak{S} - B)$.

2.25. DEFINITION. F is an ***annular*** blanket if and only if F is a blanket, and there exists such a function λ that for each x in the domain of F:

(i) $1 < \lambda(x) < \infty$;

(ii) $F(x)$ is that family of sets for which $\beta \in F(x)$ if and only if β is closed and

$$\mathfrak{C}(x, r) \subset \beta \subset \mathfrak{C}(x, r\lambda(x))$$

for some $r > 0$.

2.26. DEFINITION.

$$\mathfrak{M}\phi = \underset{x}{E} [\phi(\operatorname{sng} x) > 0].$$

Evidently $\mathfrak{M}\phi$ is a countable set of points in $\mathfrak{S}$ for each $\phi \in \mathcal{U}$.

2.27. DEFINITION.

$$\mathfrak{I}\phi = \underset{x}{E}\left[\limsup_{r \to 0+} \frac{\phi\{\mathfrak{C}(x, 2r)\}}{\phi\{\mathfrak{C}(x, r)\}} = \infty\right].$$

2.28. LEMMA. *If $\phi \in \mathcal{U}$, $\mu > 1$, and*

$$\limsup_{r \to 0+} \frac{\phi\{\mathfrak{C}(x, \mu r)\}}{\phi\{\mathfrak{C}(x, r)\}} = \infty$$

then

$$\limsup_{r\to 0+} \frac{\phi\{\mathbb{C}(x, \nu r)\}}{\phi\{\mathbb{C}(x, r)\}} = \infty$$

for each $\nu > 1$.

PROOF. Suppose, on the contrary, $\nu > 1$ and $L < \infty$ such that

$$\limsup_{r\to 0+} \frac{\phi\{\mathbb{C}(x, \nu r)\}}{\phi\{\mathbb{C}(x, r)\}} < L.$$

It follows by an inductive process that for each positive integer n,

$$\limsup_{r\to 0+} \frac{\phi\{\mathbb{C}(x, \nu^n r)\}}{\phi\{\mathbb{C}(x, r)\}} < L^n. \tag{1}$$

However, we may choose such a positive integer N that $\mu < \nu^N$, and our hypotheses imply that

$$\limsup_{r\to 0+} \frac{\phi\{\mathbb{C}(x, \nu^N r)\}}{\phi\{\mathbb{C}(x, r)\}} \geqq \limsup_{r\to 0+} \frac{\phi\{\mathbb{C}(x, \mu r)\}}{\phi\{\mathbb{C}(x, r)\}} = \infty$$

which contradicts (1) above. The lemma is therefore proved.

We note in passing the following obvious corollary.

2.29. COROLLARY. If $\phi \in \mathcal{U}$ and $\mu > 1$, then

$$\mathfrak{I}\phi = \underset{x}{E}\left[\limsup_{r\to 0+} \frac{\phi\{\mathbb{C}(x, \mu r)\}}{\phi\{\mathbb{C}(x, r)\}} = \infty\right].$$

3. **Some preliminary lemmas.** For the entirety of §3 we choose such a fixed ϕ that $\phi \in \mathcal{U}$, $\mathfrak{M}\phi = 0$, and $\phi(\mathcal{S}) < \infty$. We shall let $Q = \mathfrak{I}\phi$ for brevity.

3.1. LEMMA. *If* A *is a Borel subset of* $\mathcal{S}$, *and* $0 \leqq a \leqq 2a \leqq \phi(A)$, *then there exists a Borel set* $B \subset A$ *for which* $a \leqq \phi(B) \leqq 2a$.

PROOF. In case $a = 0$ the lemma is obviously true. Suppose then that $0 < a < \infty$. With the aid of Lindelöf's covering theorem for separable metric spaces, we observe that there exists, and we determine, a countable disjointed family of Borel sets $\beta_1, \beta_2, \beta_3, \cdots$ for which

$$\sum_{n=1}^{\infty} \beta_n = A; \qquad 0 \leqq \phi(\beta_n) \leqq a$$

for $n = 1, 2, 3, \cdots$.

Since

$$\lim_{n\to\infty}\phi\left(\sum_{m=1}^{n}\beta_m\right)=\phi\left(\sum_{m=1}^{\infty}\beta_m\right)=\phi(A)\geqq 2a>0$$

and also

$$\phi\left(\sum_{m=1}^{n}\beta_m\right)=\sum_{m=1}^{n}\phi(\beta_m)$$

for $n=1, 2, 3, \cdots$, it follows that for a suitably chosen positive integer N,

$$a\leqq\phi\left(\sum_{m=1}^{N}\beta_m\right)\leqq 2a.$$

Setting $B=\sum_{m=1}^{N}\beta_m$, the truth of the lemma is apparent.

3.2. LEMMA. *If A is a Borel subset of $\mathfrak{S}$, and $0\leqq k\leqq\phi(A)$, then there exists a Borel set $C\subset A$ for which $\phi(C)=k$.*

PROOF. With the aid Lemma 3.1 we may and do determine inductively such a disjointed sequence of Borel sets $0=B_1, B_2, B_3, \cdots$ that for each positive integer n,

$$B_{n+1}\subset\left(A-\sum_{m=1}^{n}B_m\right);$$

$$\frac{1}{2}\left\{k-\phi\left(\sum_{m=1}^{n}B_m\right)\right\}\leqq\phi(B_{n+1})\leqq k-\phi\left(\sum_{m=1}^{n}B_m\right).$$

Let $C=\sum_{m=1}^{\infty}B_m$. It is easily checked that $C\subset A$ is a Borel set for which $\phi(C)=k$. The proof of the lemma is complete.

3.3. DEFINITION. $J(B, a, b, \mu, \alpha)$ is the set of those points x such that

$$\phi\{B\cdot\mathfrak{C}(x, \mu r)\}>\alpha\phi\{\mathfrak{C}(x, r)\}$$

for some r for which $a<r<b$.

3.4. LEMMA. *If $B\subset\mathfrak{S}$, $1<\mu<\infty$, $0<\alpha<\infty$, $0\leqq a<b<\infty$, then $J(B, a, b, \mu, \alpha)$ is open.*

PROOF. We choose any point x in $J(B, a, b, \mu, \alpha)$ and 3.3 so determine r that

$$\phi\{B\cdot\mathfrak{C}(x, \mu r)\}>\alpha\phi\{\mathfrak{C}(x, r)\};\qquad a<r<b.$$

Since $\phi\in\mathcal{U}$ we determine $d>0$ so that

(1) $$\phi\{B\cdot\mathfrak{C}(x, \mu r)\}>\alpha\phi\{\mathfrak{C}(x, r+d)\};\qquad a<r+d<b.$$

Now let y be any point in $\mathfrak{C}(x, d/2)$. It is easily checked that

$$\begin{aligned} &\mathfrak{C}(x, r) \subset \mathfrak{C}(y, r + d/2) \subset \mathfrak{C}(x, r + d); \\ &\mathfrak{C}(x, \mu r) \subset \mathfrak{C}(y, \mu(r + d/2)). \end{aligned} \tag{2}$$

From (1) and (2) we note that

$$\phi\{B \cdot \mathfrak{C}(y, \mu(r + d/2))\} > \alpha\phi\{\mathfrak{C}(y, r + d/2)\}; \qquad a < r + d/2 < b.$$

Accordingly $y \in J(B, a, b, \mu, \alpha)$.

The last paragraph assures us that $\mathfrak{C}(x, d/2) \subset J(B, a, b, \mu, \alpha)$, and hence from the arbitrary nature of x in $J(B, a, b, \mu, \alpha)$, we see that $J(B, a, b, \mu, \alpha)$ is open. The lemma is proved.

3.5. LEMMA. *If $\epsilon > 0$, $1 < \mu < \infty$, $0 < \alpha < \infty$, $0 < b < \infty$, then there is a positive number a for which*

$$\phi\{Q - J(\mathfrak{S}, a, b, \mu, \alpha)\} < \epsilon; \qquad a < b; a < \epsilon.$$

PROOF. From Corollary 2.29 we note that $Q \subset J(\mathfrak{S}, 0, b, \mu, \alpha)$. The desired result follows at once with the aid of Lemma 3.4.

3.6. LEMMA. *If $\epsilon > 0$, $1 < \mu < \infty$, $1 \leqq \alpha < \infty$, $0 < b < \infty$, then there is a positive number a and a Borel set B such that*

$$\begin{aligned} &\alpha\phi(B) = \phi(\mathfrak{S}); \\ &\phi\{Q - J(B, a, b, \mu^2, 1)\} < \epsilon; \qquad a < b; a < \epsilon. \end{aligned}$$

PROOF. In accordance with Lemma 3.5, we choose such a positive number a that

$$\phi\{Q - J(\mathfrak{S}, a, b, \mu, \alpha)\} < \epsilon; \qquad a < b; a < \epsilon. \tag{1}$$

Using the fact that $\mathfrak{S}$ is a separable space we construct such a countable disjointed family T of Borel sets that $\sigma T = \mathfrak{S}$ and

$$\operatorname{diam} t < \mu(\mu - 1)a \tag{2}$$

whenever $t \in T$. We invoke Lemma 3.2 to construct such a countable disjointed family S of Borel sets in one-to-one correspondence with T that

$$\phi(s) = \frac{1}{\alpha}\phi(t); \qquad s \subset t \tag{3}$$

whenever $s \in S$ corresponds to $t \in T$.

We let

$$P = \sigma S;$$

we see that B is a Borel set and

$$\phi(B) = \sum_{s \in S} \phi(s) = \frac{1}{\alpha} \sum_{t \in T} \phi(t) = \frac{1}{\alpha} \phi(\sigma T) = \frac{1}{\alpha} \phi(\mathfrak{S}).$$

We let x be any point in $J(\mathfrak{S}, a, b, \mu, \alpha)$. We 3.3 choose a number r so that

$$\phi\{\mathfrak{C}(x, \mu r)\} > \alpha\phi\{\mathfrak{C}(x, r)\}, \qquad a < r < b, \tag{4}$$

let

$$M = \underset{t}{E}\,[t \in T \text{ and } t \cdot \mathfrak{C}(x, \mu r) \neq 0],$$

$$N = \underset{s}{E}\,[s \in S \text{ and } s \text{ corresponds to some } t \in M \text{ in the sense of (3)}],$$

and observe that M and N are countable disjointed families of Borel sets. From (2), (3), and (4) we obtain

$$\mathfrak{C}(x, \mu r) \subset \sigma M; \qquad \sigma N \subset \mathfrak{C}(x, \mu^2 r);$$

$$\phi\{B \cdot \mathfrak{C}(x, \mu^2 r)\} \geqq \phi(\sigma N) = \sum_{s \in N} \phi(s) = \frac{1}{\alpha} \sum_{t \in M} \phi(t) = \frac{1}{\alpha} \phi(\sigma M)$$

$$\geqq \frac{1}{\alpha} \phi\{\mathfrak{C}(x, \mu r)\} > \phi\{\mathfrak{C}(x, r)\}.$$

Consequently $x \in J(B, a, b, \mu^2, 1)$.

The last paragraph assures us that

$$J(\mathfrak{S}, a, b, \mu, \alpha) \subset J(B, a, b, \mu^2, 1),$$

and a glance at (1) now convinces us that

$$\phi\{Q - J(B, a, b, \mu^2, 1)\} < \epsilon; \qquad a < b; a < \epsilon.$$

The proof is complete.

3.7. LEMMA. *If F is any ϕ-free annular blanket with domain A, then* $\phi(Q \cdot A) = 0$.

PROOF. We shall prove the lemma by contradiction. We let $Q' = QA$ and suppose that $\phi(Q') > 0$. We choose any sequence of numbers $\infty > \mu_1 > \mu_2 > \mu_3 > \cdots$ for which $\lim_{k \to \infty} \mu_k = 1$. In accordance with Lemma 3.6 we determine inductively such a sequence of positive numbers $1 = a_1 > a_2 > a_3 > \cdots$ that $\lim_{k \to \infty} a_k = 0$, and a sequence of Borel sets $B_1, B_2, B_3, \cdots$ for which

$$\phi(B_k) \leqq \frac{\phi(Q')}{2^{k+1}}; \qquad \phi\{Q - J(B_k, a_{k+1}, a_k, \mu_k^2, 1)\} < \frac{\phi(Q')}{2^{k+1}} \tag{1}$$

for $k=1, 2, 3, \cdots$. We let

$$C = A \cdot \prod_{k=1}^{\infty} J(B_k, a_{k+1}, a_k, \overset{2}{\mu}_k, 1),$$

and note from (1) that

$$\phi(Q' - C) \leqq \phi\left\{Q - \prod_{k=1}^{\infty} J(B_k, a_{k+1}, a_k, \overset{2}{\mu}_k, 1)\right\} \tag{2}$$

$$< \sum_{k=1}^{\infty} \frac{\phi(Q')}{2^{k+1}} = \frac{\phi(Q')}{2}; \quad \phi(C) > \frac{\phi(Q')}{2}.$$

We let

$$B = \sum_{k=1}^{\infty} B_k,$$

note that B is a Borel set, perceive from (1) above that

$$\phi(B) \leqq \sum_{k=1}^{\infty} \phi(B_k) \leqq \sum_{k=1}^{\infty} \frac{\phi(Q')}{2^{k+1}} = \frac{\phi(Q')}{2}, \tag{3}$$

and observe from (2) and (3) that

$$\phi(C - B) > 0. \tag{4}$$

Next, we choose any point x in $C-B$ and any positive integer k. Since $x \in C$, we 3.3 determine such a positive number r that

$$\phi\{B_k \cdot \mathbb{C}(x, \overset{2}{\mu}_k r)\} > \phi\{\mathbb{C}(x, r)\}; \qquad a_{k+1} < r < a_k, \tag{5}$$

select such a closed set α that[7]

$$\phi(\alpha) > (1/2)\phi\{B_k \cdot \mathbb{C}(x, \overset{2}{\mu}_k r)\}; \qquad \alpha \subset B_k \cdot \mathbb{C}(x, \overset{2}{\mu}_k r), \tag{6}$$

and let $\beta = \mathbb{C}(x, r) + \alpha$. Clearly β is closed. From (6) we note that

$$\mathbb{C}(x, r) \subset \beta \subset \mathbb{C}(x, \overset{2}{\mu}_k r), \tag{7}$$

and from (5) and (6) we derive

$$\frac{\phi(B\beta)}{\phi(\beta)} \geqq \frac{\phi(B_k\beta)}{\phi(\beta)} \geqq \frac{\phi(\alpha)}{\phi\{\mathbb{C}(x, r)\} + \phi(\alpha)} > \frac{\phi(\alpha)}{3\phi(\alpha)} = \frac{1}{3}. \tag{8}$$

[7] See A. P. Morse, *Role of internal families in measure theory*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 723–728.

It is clear that $(C-B)\subset A$, and from 2.25, (7), (8), and the arbitrary nature of x in the last paragraph, it follows that

$$\operatorname*{limalsup}_{F(x)\exists\beta\to x}\frac{\phi(B\beta)}{\phi(\beta)}>0$$

for each x in $C-B$. Due to (4), it is 2.24 clear that F is not ϕ-free. This contradiction of our hypotheses proves the lemma.

4. Necessary and sufficient conditions that an annular blanket be ψ-strong.

4.1. THEOREM. *If F is a ψ-free annular blanket with domain A, then $A\cdot\mathfrak{J}\psi$ is of ψ-measure zero.*

PROOF. It is not difficult to see that if the theorem holds when $\psi(\mathfrak{S})<\infty$, it also holds when $\psi(\mathfrak{S})=\infty$. We therefore prove the theorem under the assumption that $\psi(\mathfrak{S})<\infty$.

We recall 2.26, and represent ψ as the sum of two measures θ and ϕ defined by

$$\theta(\beta)=\psi(\beta\cdot\mathfrak{M}\psi);\qquad \phi(\beta)=\psi(\beta)-\theta(\beta). \tag{1}$$

Since $\mathfrak{M}\psi$ is a countable set and F is 2.24 ψ-free by hypothesis, it is clear that

$$\operatorname*{limal}_{F(x)\exists\beta\to x}\frac{\theta(\beta)}{\psi(\beta)}=\operatorname*{limal}_{F(x)\exists\beta\to x}\left\{\frac{\theta(\beta)}{\phi(\beta)+\theta(\beta)}\right\}=0$$

for ψ-almost all x in $A\cdot(\mathfrak{S}-\mathfrak{M}\psi)$; from this it follows that

$$\operatorname*{limal}_{F(x)\exists\beta\to x}\frac{\theta(\beta)}{\phi(\beta)}=0 \tag{2}$$

for ψ almost all x in $A\cdot(\mathfrak{S}-\mathfrak{M}\psi)$, and hence from (1), for ϕ almost all $x\in A$.

We choose any Borel set $B\subset\mathfrak{S}$. From (2) and the fact that F is ψ-free by hypothesis, we derive

$$\operatorname*{limal}_{F(x)\exists\beta\to x}\frac{\phi(B\beta)}{\phi(\beta)}\leqq\operatorname*{limal}_{F(x)\exists\beta\to x}\frac{\psi(B\beta)}{\psi(\beta)}\cdot\operatorname*{limal}_{F(x)\exists\beta\to x}\left\{1+\frac{\theta(\beta)}{\phi(\beta)}\right\}=0$$

for ϕ almost all x in $A\cdot(\mathfrak{S}-B)$. Hence F is ϕ-free. From (1) we see that $\phi\in\mathcal{U}$, $\mathfrak{M}\phi=0$, and $\phi(\mathfrak{S})<\infty$; we may therefore apply Lemma 3.7 to conclude that

$$\phi(A\cdot\mathfrak{J}\phi)=0. \tag{3}$$

From (2) and (3) we derive

$$\limsup_{r\to 0+}\frac{\psi\{\mathfrak{C}(x,2r)\}}{\psi\{\mathfrak{C}(x,r)\}} \leqq \limsup_{r\to 0+}\frac{\psi\{\mathfrak{C}(x,2r)\}}{\phi\{\mathfrak{C}(x,r)\}}$$

$$= \limsup_{r\to 0+}\frac{\phi\{\mathfrak{C}(x,2r)\}}{\phi\{\mathfrak{C}(x,r)\}}\cdot\left\{1+\frac{\theta\{\mathfrak{C}(x,2r)\}}{\phi\{\mathfrak{C}(x,2r)\}}\right\}$$

$$= \limsup_{r\to 0+}\frac{\phi\{\mathfrak{C}(x,2r)\}}{\phi\{\mathfrak{C}(x,r)\}} < \infty$$

for ϕ almost all x in A. It is clear that

$$\limsup_{r\to 0+}\frac{\psi\{\mathfrak{C}(x,2r)\}}{\psi\{\mathfrak{C}(x,r)\}} < \infty \tag{4}$$

for each $x\in\mathfrak{M}\psi$. It is therefore apparent that (4) holds for ψ almost *all* $x\in A$. Thus we 2.27 observe that $A\cdot\mathfrak{F}\psi$ is of ψ-measure zero. The theorem is proved.

4.2. THEOREM. *If F is a ψ-free annular blanket, then F is ψ-diametrically regular.*

PROOF. The desired conclusion follows from 4.1, 2.25, 2.29, and reference to Definition 2.13.

4.3. THEOREM. *A necessary and sufficient condition that an annular blanket F be ψ-strong is that it be ψ-free.*

PROOF. We check the sufficiency first. If F is ψ-free it is 4.2 ψ-diametrically regular, and hence is 2.21 ψ-strong.

We now prove the necessity by contradiction. We suppose F to be ψ-strong yet not ψ-free. With the aid of 2.20 and 2.22 we determine a bounded Borel set B, a bounded set $D\subset(\mathfrak{S}-B)$ of positive ψ-measure, and a number d for which

$$\operatorname*{limal}_{F(x)\ni\beta\to x}\frac{\psi(B\beta)}{\psi(\beta)} > d > 0$$

whenever $x\in D$.

We select such a closed set $C\subset B$ that[8]

$$\psi(B-C) < d\psi(D), \tag{1}$$

choose a bounded open set S containing $B+D$, let

$$G(x) = F(x)\cdot\underset{\beta}{E}\left[\beta\subset(S-C);\frac{\psi(B\beta)}{\psi(\beta)} > d\right], \tag{2}$$

[8] See footnote 7.

for each $x \in D$, and observe that G is a subblanket of F whose domain is D. Thus G is 2.18 ψ-heavy. We 2.16 extract from the spread of G such a countable disjointed family τ that

$$\psi(D - \sigma\tau) = 0, \tag{3}$$

and from (2) and (3) observe that

$$0 < \psi(D) \leqq \psi(\sigma\tau) = \sum_{\beta \in \tau} \psi(\beta) < \frac{1}{d} \sum_{\beta \in \tau} \psi(B\beta)$$

$$= \frac{1}{d}\, \psi(B\sigma\tau) \leqq \frac{1}{d}\, \psi(B - C).$$

This last result contradicts (1) and proves the theorem.

5. **A nearly spherical, star-shaped blanket which is not universal.** In this section we shall construct the example mentioned in the introduction. For the remainder of the paper we shall let $\mathfrak{S}$ be the euclidean plane, ρ the usual metric. We shall let L denote Carathéodory linear measure in $\mathfrak{S}$.

We now proceed to define a special set[9] in $\mathfrak{S}$ which plays a basic role in the ensuing work. We shall hereinafter call a set in $\mathfrak{S}$ a *triangle* if and only if it is a closed set of points consisting of the interior and boundary of an equilateral triangle with its base line horizontal and below the opposite vertex. Thus the length of each side of a triangle is equal to its diameter.

For T a triangle and n a positive integer, we construct $2n+1$ triangles, each with diameter equal to

$$\frac{\text{diam } T}{2n+1},$$

so placed that the union of their bases is the base of T; next, we take the middle triangle and translate it vertically upward until its top vertex coincides with the top vertex of T; we denote the $2n+1$ triangles so obtained by $O_n(T)$. We let T_0 be a triangle of diameter 1. We put $\mathfrak{F}_0 = \text{sng } T_0$; we so define $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \cdots$ that

$$\mathfrak{F}_n = \sum_{T \in \mathfrak{F}_{n-1}} O_n(T)$$

for each positive integer n.

We see that $\sigma\mathfrak{F}_1 \supset \sigma\mathfrak{F}_2 \supset \sigma\mathfrak{F}_3 \cdots$ and define

$$\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 + \cdots;$$

[9] See pp. 285–286 of RM.

$$\mathcal{A} = \prod_{n=1}^{\infty} \sigma\mathfrak{F}_n.$$

Since $T \in \mathfrak{F}$ implies that the projection of $\mathcal{A}T$ on the base of T is the base of T, we have little difficulty in verifying that $L(\mathcal{A}T) = \text{diam } T$ for $T \in \mathfrak{F}$. In particular, $L(\mathcal{A}) = 1$.

We shall term a triangle in $\mathfrak{F}$ an *apex* triangle if and only if its upper vertex coincides with the top vertex of some larger triangle in $\mathfrak{F}$. We let W be the set so defined that $x \in W$ if and only if $x \in \mathcal{A}$ and the number of apex triangles to which x belongs is finite. Since

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{2n+1}\right) = 0,$$

we conclude that $L(W) = 0$. We let $V = \mathcal{A} - W$ and see that $L(V) = L(\mathcal{A}) = 1$.

If x is any point in V, there clearly exists a countable family $\mathfrak{G}$ of triangles whose diameters tend to zero, such that whenever $T \in \mathfrak{G}$ we can find such a triangle T' and such a positive integer n that $T \in O_n(T')$; T is an apex triangle; $x \in T$. Let $r = 1/3 \text{ diam } T'$. It is easily seen that $\mathfrak{C}(x, r)$ contains T, and contains no points of other triangles belonging to $O_n(T')$; $\mathfrak{C}(x, 3r)$ contains the whole triangle T'. Thus

$$\frac{L(\mathcal{A} \cdot \mathfrak{C}(x, 3r))}{L(\mathcal{A} \cdot \mathfrak{C}(x, r))} = 2n + 1.$$

We see from this that

$$\limsup_{r \to 0+} \frac{L(\mathcal{A} \cdot \mathfrak{C}(x, 3r))}{L(\mathcal{A} \cdot \mathfrak{C}(x, r))} = \infty$$

for each $x \in V$.

Hereinafter we shall let ϕ denote that measure for which $\phi(\beta) = L(\mathcal{A}\beta)$ for each $\beta \subset \mathfrak{S}$. From the preceding remarks we see that $V \subset \mathfrak{I}\phi$ and thus $\phi(\mathfrak{I}\phi) = 1$.

5.1. DEFINITION. We say that a set of points $\beta \subset \mathfrak{S}$ is *star-shaped about* x if and only if $x \in \beta$, and every point of the closed line segment joining y to x belongs to β whenever $y \in \beta$.

5.2. DEFINITION. We say that G is a *star-shaped* blanket if and only if G is such a blanket that all members of $G(x)$ are star-shaped about x whenever x is in the domain of G.

5.3. DEFINITION. We define S as that function for which $S(x, \beta)$ is the set of points comprising the union of those straight line seg-

ments joining x to the points of β.

Evidently $S(x, \beta)$ is star-shaped about x for each $x \in \mathcal{S}$, $\beta \subset \mathcal{S}$.

5.4. DEFINITION. We term *internal radius of β at x* the sup of numbers r for which $\mathbb{C}(x, r) \subset \beta$.

5.5. DEFINITION. We define g as that function for which

$$g(r) = \inf_{x \in \mathcal{A}} \phi\{\mathbb{C}(x, r)\}.$$

5.6. LEMMA. *$g(r) > 0$ whenever $r > 0$.*

This is easily verified by routine methods.

5.7. LEMMA. *For each $\epsilon > 0$ there exists such a number $r > 0$ that $\phi\{S(x, \mathbb{C}(y, r))\} \leqq \epsilon$ whenever $x \in \mathcal{A}$, $y \in \mathcal{S}$.*

PROOF. We choose $\epsilon > 0$. It is easily seen that $\phi\{\mathbb{C}(x, \epsilon/2)\} = L\{\mathcal{A} \cdot \mathbb{C}(x, \epsilon/2)\} \leqq \text{diam } \mathbb{C}(x, \epsilon/2) = \epsilon$ for each $x \in \mathcal{S}$. It is also known[10] that any straight line in the plane may intersect $\mathcal{A}$ in at most a set of ϕ-measure zero. Due to this, we may and do determine such a number $b > 0$ that each circular sector of central angle less than b, whose center is a point of $\mathcal{A}$, is of ϕ-measure less than ϵ.

We further determine such a number $r > 0$ that for each $x \in \mathcal{S}$, $y \in \mathcal{S}$,

(i) $\mathcal{S}\{x, \mathbb{C}(y, r)\}$ is contained in a sector of central angle less than b whenever $\rho(x, y) \geqq \epsilon/4$;

(ii) $\mathcal{S}\{x, \mathbb{C}(y, r)\} \subset \mathbb{C}(x, \epsilon/2)$ whenever $\rho(x, y) < \epsilon/4$.

From the preceding two paragraphs we conclude that $\phi\{\mathcal{S}(x, \mathbb{C}(y, r))\} \leqq \epsilon$ whenever $x \in \mathcal{A}$, $y \in \mathcal{S}$. The proof of the lemma is complete.

We recall Definition 3.3, and prove the following lemma.

5.8. LEMMA. *If $0 < a < b < \infty$, $1 < \mu < \infty$, $0 < \alpha < \infty$, then there is such a Borel set $B \subset \mathcal{S}$ that for each $x \in \mathcal{A} \cdot \{J(\mathcal{S}, a, b, \mu, \alpha) - B\}$ there exist some r and β for which:*

(i) *β is closed and star-shaped about x;*

(ii) $\mathbb{C}(x, r) \subset \beta \subset \mathbb{C}(x, \mu^2 r)$; $a < r < b$;

(iii) $\phi\{\mathbb{C}(x, r)\} < L(B\beta) \leqq L(B) = 1/\alpha$;

(iv) $\phi(\beta) < 2\phi\{\mathbb{C}(x, r)\}$;

(v) $\phi(B) = 0$.

PROOF. We choose such a disjointed family T of squares of equal size, half open to the right and on top, that $\sigma T = \mathcal{S}$ and

[10] In fact, if C denotes any strictly rectifiable curve in $\mathcal{S}$, then $L(\mathcal{A} \cdot C) = 0$. This may be established by the following references in RM: 2.13, 2.16, 11.1 (1) and (4), and the remark in the middle of page 286 that $\mathcal{A}$ is not L-rectifiable.

$$(1)\quad \operatorname{diam} t < \mu(\mu-1)a$$

whenever $t \in T$; we denote by M the family of all sets β of the form $\beta = \mathfrak{C}(x, \mu r)$ for which

$$(2)\quad \phi\{\mathfrak{C}(x, \mu r)\} > \alpha\phi\{\mathfrak{C}(x, r)\}; \qquad a < r < b;$$

and we let

$$T' = T \cdot \underset{t}{E} [t \cdot \sigma M \neq 0; \phi(t) > 0].$$

It is clear that T' is a finite subset of T. We let N denote the number of members of T'.

For each $t \in T'$, we proceed as follows. With the aid of 5.6 and 5.7, we construct a circular disk Δ_t whose center coincides with that of t, of radius so small that

$$(3)\quad \Delta_t \subset t; \qquad \phi\{S(x, \Delta_t)\} \leqq \frac{g(a)}{N+1}$$

for each $x \in \mathcal{A}$. Next we construct such a countable family $\mathfrak{Z}_t$ of concentric circles that

$$(4)\quad L(\mathfrak{Z}_t) = \frac{1}{\alpha}\phi(t) > 0; \qquad \sigma\mathfrak{Z}_t \subset \Delta_t;$$

and we let

$$(5)\quad B = \sum_{t \in T'} \sigma\mathfrak{Z}_t.$$

Clearly B is a Borel set; it is the totality of points belonging to an enumerable family of circles. For this reason[11]

$$(6)\quad \phi(B) = L(B\mathcal{A}) = 0,$$

which confirms (v) above.

We let $Q' = \mathcal{A} \cdot \{J(\mathfrak{S}, a, b, \mu, \alpha) - B\}$, choose any point x in Q', determine such an r that (2) holds, let

$$T'' = T' \cdot \underset{t}{E} [t \cdot \mathfrak{C}(x, \mu r) \neq 0],$$

and put

$$\beta = \mathfrak{C}(x, r) + \sum_{t \in T''} S(x, \Delta_t).$$

Since T'' has at most finitely many members, it follows that β is

[11] See footnote 10.

closed. It is likewise obvious that β is star-shaped about x. This confirms (i) above.

From (1), (2), and (3) we see that

$$\Delta_t \subset t \subset \mathfrak{C}(x, \mu^2 r); \qquad S(x, \Delta_t) \subset \mathfrak{C}(x, \mu^2 r)$$

for each $t \in T''$, and hence

$$\sum_{t \in T''} S(x, \Delta_t) \subset \mathfrak{C}(x, \mu^2 r).$$

We therefore see that

$$\mathfrak{C}(x, r) \subset \beta \subset \mathfrak{C}(x, \mu^2 r),$$

thereby confirming (ii).

We next observe that

$$(7) \quad L(B) = \sum_{t \in T'} L(\sigma \mathfrak{Z}_t) = \frac{1}{\alpha} \sum_{t \in T'} \phi(t) = \frac{1}{\alpha} \phi(\sigma T') = \frac{1}{\alpha} \phi(\mathfrak{S}) = \frac{1}{\alpha},$$

and that

$$L(B\beta) \geqq L\left\{ \sum_{t \in T''} B \cdot S(x, \Delta_t) \right\} \geqq L\left\{ \sum_{t \in T''} B\Delta_t \right\}$$

$$(8) \qquad = L\left\{ \sum_{t \in T''} \sigma \mathfrak{Z}_t \right\} = \sum_{t \in T''} L(\sigma \mathfrak{Z}_t) = \frac{1}{\alpha} \sum_{t \in T''} \phi(t)$$

$$= \frac{1}{\alpha} \phi(\sigma T'').$$

It follows from the definition of T'' that

$$(9) \qquad \phi\{\mathfrak{C}(x, \mu r) - \sigma T''\} = 0.$$

Accordingly from (2), (8), and (9) we derive

$$L(B\beta) \geqq \frac{1}{\alpha} \phi(\sigma T'') \geqq \frac{1}{\alpha} \phi\{\mathfrak{C}(x, \mu r)\} > \phi\{\mathfrak{C}(x, r)\},$$

which, together with (7), completes the confirmation of (iii).

Finally, from (3) and Definition 5.5 we obtain

$$\phi(\beta) \leqq \phi\{\mathfrak{C}(x, r)\} + \sum_{t \in T''} \phi\{S(x, \Delta_t)\} \leqq \phi\{\mathfrak{C}(x, r)\} + \frac{N}{N+1} g(a)$$

$$< \phi\{\mathfrak{C}(x, r)\} + g(a) \leqq 2\phi\{\mathfrak{C}(x, r)\},$$

which confirms (iv) and completes the proof of the lemma.

5.9. THEOREM. *There exists a Borel set $B_0 \subset \mathfrak{S}$, a measure $\Omega \in \mathcal{U}$, and a close star-shaped blanket G, for which*

(i) $\Omega(\beta) = L(B_0\beta)$ *for each* $\beta \subset \mathfrak{S}$;

(ii) G *is not* Ω*-free*;

(iii) *for each* x *in the domain of* G

$$\operatorname*{limal}_{G(x)\ni\beta\to x} (\operatorname{diam}\beta)\,(\textit{internal radius of } \beta \textit{ at } x)^{-1} = 2.$$

PROOF. We let $Q = \mathfrak{I}\phi$. We select any sequence of numbers $\infty > \mu_1 > \mu_2 > \mu_3 > \cdots$ for which $\lim_{k\to\infty} \mu_k = 1$. Since $\phi \in \mathcal{U}$, $\mathfrak{M}\phi = 0$, and $\phi(\mathfrak{S}) < \infty$, we may employ Lemma 3.5 to define inductively such a sequence of numbers $1 = a_1 > a_2 > a_3 > \cdots$ that $\lim_{k\to\infty} a_k = 0$, and for which

$$\phi\{Q - J(\mathfrak{S}, a_{k+1}, a_k, \mu_k, 2^k)\} < \frac{1}{2^{k+1}}$$

for $k = 1, 2, 3, \cdots$.

For brevity, we let

$$D_k = J(\mathfrak{S}, a_{k+1}, a_k, \mu_k, 2^k)$$

and

$$D = \mathcal{A} \cdot \prod_{k=1}^{\infty} D_k.$$

Accordingly we observe

$$\phi\left\{Q - \prod_{k=1}^{\infty} D_k\right\} \leqq \sum_{k=1}^{\infty} \phi(Q - D_k) \leqq \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2},$$

recall that $\phi(Q) = 1$, and note that

$$\begin{aligned} \phi(D) &= \phi\left(\prod_{k=1}^{\infty} D_k\right) \geqq \phi(Q) - \phi\left(Q - \prod_{k=1}^{\infty} D_k\right) \\ &\geqq 1 - \frac{1}{2} = \frac{1}{2} > 0. \end{aligned} \tag{1}$$

With the aid of Lemma 5.8 we construct such a sequence of Borel sets $B_1, B_2, B_3, \cdots$ that for each $x \in \mathcal{A} \cdot (D_k - B_k)$ we may find some r and β for which

(2) β is closed and star-shaped about x;

(3) $\mathfrak{C}(x, r) \subset \beta \subset \mathfrak{C}(x, \mu_k r)$; $\quad a_{k+1} < r < a_k$;

$$\phi\{\mathfrak{C}(x, r)\} < L(B_k\beta) \leqq L(B_k) = \frac{1}{2^k}; \tag{4}$$

$$\phi(\beta) < 2\phi\{\mathfrak{C}(x, r)\}; \tag{5}$$

$$\phi(B_k) = 0. \tag{6}$$

We let

$$B = \sum_{k=1}^{\infty} B_k,$$

observe that B is a Borel set, note from (6) that $\phi(B)=0$, let $B_0=B+\mathcal{A}$, and define

$$\Omega(\beta) = \phi(\beta) + L(B\beta) = L(\mathcal{A}\beta) + L(B\beta) = L\{(\mathcal{A} + B)\beta\} = L(B_0\beta)$$

for each $\beta \subset \mathfrak{S}$.

From (4) we find

$$L(B) \leqq \sum_{k=1}^{\infty} L(B_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

and hence

$$\Omega(\mathfrak{S}) = \phi(\mathfrak{S}) + L(B) \leqq 2,$$

which shows that $\Omega \in \mathcal{U}$ and completes the confirmation of (i).

From (1) we see that

$$\Omega(D - B) = \phi(D - B) = \phi(D) \geqq 1/2 > 0. \tag{7}$$

We choose any x in $D-B$ and any positive integer k, determine an r and a β for which (2), (3), (4), and (5) hold, and observe that

$$\frac{\Omega(B\beta)}{\Omega(\beta)} \geqq \frac{L(B\beta)}{\phi(\beta) + L(B\beta)} \geqq \frac{L(B_k\beta)}{\phi(\beta) + L(B_k\beta)} > \frac{L(B_k\beta)}{3L(B_k\beta)} = \frac{1}{3} > 0. \tag{8}$$

We may therefore define a close, star-shaped blanket G of domain $D-B$, which, by virtue of (3), satisfies (iii). It is clear from (7) and (8) that G is not Ω-free, which confirms (ii) and completes the proof of the theorem.

5.10. COROLLARY. *There exists a close, star-shaped blanket F for which*

(i) $\operatorname{limal}_{F(x)\ni\beta\to x}$ (diam β) (*internal radius of β at x*)$^{-1}=2$.

(ii) *F is not universal.*

PROOF. We construct Ω and G as in Theorem 5.9. We further select any blanket H for which domain H = domain G, and β is a closed circle

centered at x whenever $\beta \in H(x)$. We let F be that blanket for which

$$F(x) = G(x) + H(x)$$

for each $x \in$(domain G). It is clear from (iii) of 5.9 that F is a close star-shaped blanket for which

$$\operatorname*{limal}_{F(x) \ni \beta \to x} (\text{diam } \beta)\,(\text{internal radius of } \beta \text{ at } x)^{-1} = 2$$

for each $x \in$(domain G). This confirms (i) above.

It is known[13] that H is a 2.19 perfect blanket and is therefore Ω-strong. It is easy to see by the method of proof used in Theorem 4.3 that H is Ω-free. However, we know that G is not Ω-free; hence we 2.24 determine a Borel set B and a set $D \subset (\mathbb{S} - B)$ for which $\Omega(D) > 0$ and

$$\operatorname*{limal}_{H(x) \ni \beta \to x} \frac{\Omega(B\beta)}{\Omega(\beta)} = 0; \qquad \operatorname*{limal}_{G(x) \ni \beta \to x} \frac{\Omega(B\beta)}{\Omega(\beta)} \neq 0$$

for each $x \in D$. It is therefore clear from 2.20 and 2.23 that F is not universal. The proof is complete.

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[13] See footnote 6.

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